2 Fourier series - finding the coefficients

Recall that to solve the Dirichlet and Neumann problems for the heat and wave equations on the finite interval $(0, l)$, we need to find respectively:

i) the Fourier sine series expansion of a function $\phi(x)$ (representing the initial data) on the interval $(0, l)$,

$$\phi(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l}. \tag{1}$$

ii) the Fourier cosine series expansion of $\phi(x)$ on $(0, l)$,

$$\phi(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{l}. \tag{2}$$

Leaving the question of whether such an infinite series is well-defined and in which sense $\phi(x)$ is equivalent to its Fourier series for later lectures, let us assume that $\phi(x)$ can be written as such series, and that the Fourier series in question do converge to the function $\phi(x)$ in a suitable sense. In order to use the Fourier expansions in solving boundary value problems with the separation of variables method, we need to find the coefficients A_n in the expansions (1) and (2).

Let us start with the sine expansion (1). Notice that $\phi(x)$ is written as a linear combination of the functions

$$\left\{ \sin \frac{n\pi x}{l} \right\}_{n=1}^{\infty}, \tag{3}$$

which is similar to expansions of vectors in terms of a basis. Recall that if $\{\mathbf{e_1}, \mathbf{e_2}, \dots, \mathbf{e_k}\}$ forms an orthonormal basis, and the vector $\mathbf{v}$ is written as a linear combination of these basis elements,

$$\mathbf{v} = c_1\mathbf{e_1} + c_2\mathbf{e_2} + \cdots + c_k\mathbf{e_k},$$

then the components of the vector $\mathbf{v} = (c_1, c_2, \dots c_k)$ can be found by projecting the vector $\mathbf{v}$ onto the respective directions $\mathbf{e_1}, \mathbf{e_2}, \dots, \mathbf{e_k}$. This is done by taking the dot (scalar, inner) product of the vector $\mathbf{v}$ with the corresponding basis element.

$$\mathbf{v} \cdot \mathbf{e_i} = \sum_j c_j \mathbf{e_j} \cdot \mathbf{e_i} = c_i,$$

since $\mathbf{e_j} \cdot \mathbf{e_i} = 1$, if $i = j$, and zero otherwise due to the orthonormality of the basis. Thus, $c_i = \mathbf{v} \cdot \mathbf{e_i}$.

We would like to compute the Fourier coefficients in a similar way. But in order for this procedure to work for expansion (1), the elements in the set (3) must be pairwise orthogonal in an appropriate sense. This turns out to be the case, if one takes the dot product of two functions f, g defined on the interval $(0, l)$ to be

$$(f, g) = \int_0^l f(x)g(x)\, dx. \tag{4}$$

This definition gives a proper inner product, and one has the associated orthogonality notion: f and g are orthogonal to each other, if their dot product is zero, $(f, g) = 0$.

Let us now check that the set (3) is indeed orthogonal, i.e. consists of pairwise orthogonal elements in the sense of the above defined dot product. That is, we want to show that

$$\int_0^l \sin \frac{n\pi x}{l} \sin \frac{m\pi x}{l}\, dx = 0, \quad \text{if } m \neq n. \tag{5}$$

Using the trigonometric identity for the product of two sines,

$$\sin\alpha\sin\beta = \frac{1}{2}\cos(\alpha-\beta) - \frac{1}{2}\cos(\alpha+\beta), \tag{6}$$

we can compute the above integral as follows:

$$\int_0^l \sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l}\,dx = \int_0^l \left(\frac{1}{2}\cos\frac{(n-m)\pi x}{l} - \frac{1}{2}\cos\frac{(n+m)\pi x}{l}\right)dx$$
$$= \frac{l}{2(n-m)\pi}\sin\frac{(n-m)\pi x}{l}\Big|_0^l - \frac{l}{2(n+m)\pi}\sin\frac{(n+m)\pi x}{l}\Big|_0^l = 0,$$

since sine vanishes for any multiple of π. Thus, (5) holds, and we can find the coefficients in the expansion (1) just as was done for regular vectors. For this, we form the dot product of the function $\phi(x)$ with the functions $\sin(m\pi x/l)$ for $m = 1, 2, \dots$, and integrate the resulting series term by term to get

$$\int_0^l \phi(x)\sin\frac{m\pi x}{l}\,dx = \int_0^l \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l}\,dx$$
$$= \sum_{n=1}^{\infty} A_n \int_0^l \sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l}\,dx = A_m\int_0^l \sin^2\frac{m\pi x}{l}\,dx,$$

since, due to the pairwise orthogonality, only the term with $n = m$ survives in the sum. For the last integral we use the trigonometric identity (6) with $n = m$ to compute

$$\int_0^l \sin^2\frac{m\pi x}{l}\,dx = \int_0^l \left(\frac{1}{2} - \frac{1}{2}\cos\frac{2m\pi x}{l}\right)dx = \frac{l}{2}.$$

Substituting this into the previous computation, we get

$$\int_0^l \phi(x)\sin\frac{m\pi x}{l}\,dx = A_m\frac{l}{2},$$

or

$$A_m = \frac{2}{l}\int_0^l \phi(x)\sin\frac{m\pi x}{l}\,dx. \tag{7}$$

This means that if $\phi(x)$ can be written in the form (1), then the coefficients of this expansion are necessarily given by (7).

We can compute the coefficients of the Fourier cosine series (2) in a similar way. For this, we need to guarantee that the set

$$\left\{\cos\frac{n\pi x}{l}\right\}_{n=0}^{\infty} = \left\{1, \cos\frac{n\pi x}{l} : n = 1, 2, \dots\right\} \tag{8}$$

is orthogonal as well. To show this, we employ the trigonometric identity for the product of cosines,

$$\cos\alpha\cos\beta = \frac{1}{2}\cos(\alpha-\beta) + \frac{1}{2}\cos(\alpha+\beta). \tag{9}$$

We thus have for $m \neq n$,

$$\int_0^l \cos\frac{n\pi x}{l}\cos\frac{m\pi x}{l}\,dx = \int_0^l \left(\frac{1}{2}\cos\frac{(n-m)\pi x}{l} + \frac{1}{2}\cos\frac{(n+m)\pi x}{l}\right)dx = 0,$$

while when $n = m$,

$$\int_0^l \cos^2 \frac{m\pi x}{l}\, dx = \int_0^l \left(\frac{1}{2} + \frac{1}{2} \cos \frac{2m\pi x}{l} \right) dx = \frac{l}{2}.$$

Then taking the dot product of $\phi(x)$ with $\cos(m\pi x/l)$ for $m = 1, 2, \dots$, and using expansion (2), we get

$$\begin{aligned}\int_0^l \phi(x) \cos \frac{m\pi x}{l}\, dx &= \int_0^l \left(\frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{l} \right) \cos \frac{m\pi x}{l}\, dx \\ &= \frac{A_0}{2} \int_0^l \cos \frac{m\pi x}{l}\, dx + \sum_{n=1}^{\infty} A_n \int_0^l \cos \frac{n\pi x}{l} \cos \frac{m\pi x}{l}\, dx = A_m \int_0^l \cos^2 \frac{m\pi x}{l}\, dx = A_m \frac{l}{2}.\end{aligned}$$

For $m = 0$, $\cos(m\pi x/l) = 1$, and hence,

$$\int_0^l \phi(x)\, dx = \frac{A_0}{2} \int_0^l dx + \sum_{n=1}^{\infty} A_n \int_0^l \cos \frac{n\pi x}{l} \cos \frac{0\pi x}{l}\, dx = \frac{A_0}{2} l.$$

So for all $n = 0, 1, 2, \dots$, we have the following formula for the coefficients of the Fourier cosine series (2)

$$A_n = \frac{2}{l} \int_0^l \phi(x) \cos \frac{n\pi x}{l}\, dx. \tag{10}$$

Similar to the case of the sine series, this formula means that if a function can be written in the form (2), then the coefficients of this expansion must be necessarily given by formula (10). Notice that the coefficient A_0 can be computed by the same formula (10), which is exactly the reason we used the factor of $1/2$ in front of the A_0 term in expansion (2).

2.1 Application to heat and wave problems

Let us now go back to the boundary value problems for the heat and wave equations, and see how the coefficients formulas allow one to find the series solution through the method of separation of variables. As an example, we will look at the Dirichlet problem for the wave equation,

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \quad \text{for } 0 < x < l, \\ u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x), \\ u(0,t) = u(l,t) = 0. \end{cases} \tag{11}$$

As we know, the series solution for the above Dirichlet problem is

$$u(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos \frac{n\pi ct}{l} + B_n \sin \frac{n\pi ct}{l} \right) \sin \frac{n\pi x}{l}, \tag{12}$$

provided the initial data can be expanded into the series

$$\phi(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l}, \qquad \psi(x) = \sum_{n=1}^{\infty} \frac{n\pi c}{l} B_n \sin \frac{n\pi x}{l}.$$

But then from our previous discussion, we know that A_n, and $(n\pi c/l)B_n$ are given by the Fourier sine series coefficients formula (7). Hence,

$$A_n = \frac{2}{l} \int_0^l \phi(x) \sin \frac{n\pi x}{l}\, dx, \qquad B_n = \frac{2}{n\pi c} \int_0^l \psi(x) \sin \frac{n\pi x}{l}\, dx.$$

Computing these values of the coefficients A_n and B_n for $n = 1, 2, \ldots$ from the given initial data, and plugging them into the series solution (12) will yield the solution to the Dirichlet wave problem (11).

One proceeds in exactly the same way for the case of Neumann problems, where instead of sine series, one works with cosine series. We demonstrate this on the example of Neumann heat problem,

$$\begin{cases} u_t - ku_{xx} = 0 \quad \text{for } 0 < x < l, \\ u(x, 0) = \phi(x), \\ u_x(0, t) = u_x(l, t) = 0. \end{cases} \tag{13}$$

Recall that the series solution is

$$u(x, t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} e^{-(n\pi/l)^2 kt} \cos \frac{n\pi x}{l}, \tag{14}$$

provided the initial data can be expanded into the series

$$\phi(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{l}.$$

The coefficients A_n can be computed from the given initial data using the Fourier cosine series coefficients formula (10). One then plugs the values of these coefficients into the series solution (14), arriving at the solution for the Neuman heat problem (13).

2.2 Examples of computing Fourier series

Let us now consider several examples of computing the coefficients in Fourier expansions for particular functions, using the coefficients formulas (7) and (10).

Example 2.1. Compute the Fourier sine and cosine series for the function $\phi(x) \equiv 1$ on the interval $(0, l)$.

To find the sine series for this function, we use the coefficients formula (7).

$$A_n = \frac{2}{l} \int_0^l 1 \cdot \sin \frac{n\pi x}{l} \, dx = -\frac{2}{l} \frac{l}{n\pi} \cos \frac{n\pi x}{l} \bigg|_0^l = \frac{2}{n\pi}(1 - \cos n\pi) = \begin{cases} 4/(n\pi) & \text{for } n - \text{odd}, \\ 0 & \text{for } n - \text{even}. \end{cases}$$

So the Fourier sine series of the function $\phi(x) \equiv 1$ is

$$1 = \frac{4}{\pi} \sin \frac{n\pi x}{l} + \frac{4}{3\pi} \sin \frac{3\pi x}{l} + \frac{4}{5\pi} \sin \frac{5\pi x}{l} + \cdots = \frac{4}{\pi} \sum_{n-\text{odd}} \frac{1}{n} \sin \frac{n\pi x}{l}.$$

To find the cosine series, we use formula (10), and compute for $n = 1, 2, \ldots$

$$A_n = \frac{2}{l} \int_0^l 1 \cdot \cos \frac{n\pi x}{l} \, dx = \frac{2}{l} \frac{l}{n\pi} \sin \frac{n\pi x}{l} \bigg|_0^l = 0,$$

while for $n = 0$ we have

$$A_n = \frac{2}{l} \int_0^l 1 \cdot \cos 0 \, dx = \frac{2}{l} l = 2.$$

So the Fourier cosine expansion of the function $\phi(x) \equiv 1$ is

$$1 = \frac{2}{2} + 0 \cdot \cos \frac{\pi x}{l} + 0 \cdot \cos \frac{2\pi x}{l} + \cdots = 1.$$

In other words, the constant function 1 is its own Fourier series, which is not surprising, taking into account the fact that the constant 1 function is an element of the set (8). □

Example 2.2. Compute the Fourier sine and cosine series for the function $\phi(x) = x$ on the interval $(0, l)$.

Using the formulas for the coefficients of the sine series and integrating by parts, we compute

$$A_n = \frac{2}{l}\int_0^l x \sin\frac{n\pi x}{l}\,dx = -\frac{2}{l}\frac{l}{n\pi}x\cos\frac{n\pi x}{l}\bigg|_0^l + \frac{2}{n\pi}\int_0^l \cos\frac{n\pi x}{l}\,dx = -\frac{2l}{n\pi}\cos n\pi = (-1)^{n+1}\frac{2l}{n\pi}.$$

So the Fourier sine series of $\phi(x) = x$ is

$$x = \frac{2l}{\pi}\sin\frac{\pi x}{l} - \frac{2l}{2\pi}\sin\frac{2\pi x}{l} + \frac{2l}{3\pi}\sin\frac{3\pi x}{l} - \cdots = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{2l}{n\pi}\sin\frac{n\pi x}{l}. \tag{15}$$

From the cosine series coefficients formula, and again integrating by parts, we get for $n = 1, 2, \ldots$

$$\begin{aligned} A_n = \frac{2}{l}\int_0^l x\cos\frac{n\pi x}{l}\,dx &= \frac{2}{n\pi}x\sin\frac{n\pi x}{l}\bigg|_0^l - \frac{2}{n\pi}\int_0^l \sin\frac{n\pi x}{l}\,dx \\ &= \frac{2l}{n^2\pi^2}\cos\frac{n\pi x}{l}\bigg|_0^l = \begin{cases} -4l/(n^2\pi^2) & \text{for } n - \text{odd}, \\ 0 & \text{for } n - \text{even}. \end{cases} \end{aligned}$$

For $n = 0$, we have

$$A_0 = \frac{2}{l}\int_0^l x\,dx = \frac{2}{l}\frac{x^2}{2}\bigg|_0^l = l.$$

So the Fourier cosine series of the function $\phi(x) = x$ is

$$x = \frac{l}{2} - \frac{4l}{\pi^2}\cos\frac{\pi x}{l} - \frac{4l}{3^2\pi^2}\cos\frac{3\pi x}{l} - \cdots = \frac{l}{2} - \sum_{n-\text{odd}}\frac{4l}{n^2\pi^2}\cos\frac{n\pi x}{l}.$$

□

Example 2.3. Find the Fourier cosine series of the function $\phi(x) = x^2$ by integration of the sine series of the function x. Assume that one can integrate the Fourier sine series of x term by term.

Integrating both sides of identity (15), we get

$$\frac{x^2}{2} = c + \sum_{n=1}^{\infty}(-1)^{n+1}\frac{2l}{n\pi}\int \sin\frac{n\pi x}{l}\,dx = c + \sum_{n=1}^{\infty}(-1)^n\frac{2l^2}{n^2\pi^2}\cos\frac{n\pi x}{l},$$

where c is the constant of integration. Thus, we get

$$x^2 = 2c + \sum_{n=1}^{\infty}(-1)^n\frac{4l^2}{n^2\pi^2}\cos\frac{n\pi x}{l}.$$

From the coefficients formula for the Fourier cosine series, we know that the free term $2c$ must equal $A_0/2$, where A_0 can be computed directly from the coefficients formula (10) as follows

$$A_0 = \frac{2}{l}\int_0^l x^2\cos 0\,dx = \frac{2}{l}\frac{x^3}{3}\bigg|_0^l = \frac{2l^2}{3}.$$

Then $2c = A_0/2 = l^2/3$, and substituting this into the above series yields the Fourier cosine series of the function x^2.

$$x^2 = \frac{l^2}{3} + \sum_{n=1}^{\infty}(-1)^n\frac{4l^2}{n^2\pi^2}\cos\frac{n\pi x}{l}.$$

One could, of course, compute the coefficients in the Fourier expansion of x^2 directly from the formula (10), however it requires integrating by parts twice, which is a little more involved than the above procedure. On the other hand, the idea of integrating (or differentiating) Fourier series term by term can yield Fourier series for new functions by a fairly simple computation. □

2.3 Conclusion

Using the method of separation of variables for the heat and wave boundary value problems we derived the series solutions, in which the coefficients came from the sine and cosine expansions of the initial data. In this lecture we developed a method of computing these coefficients, arriving at formulas (7) and (10) respectively, thus providing a means of computing the solutions to the boundary value problems completely. One should keep in mind, however, that in all of our computations we assumed that the series expansion is well defined, i.e. the series converges in an appropriate sense. We will return to the question of convergence of Fourier series in subsequent lectures, and put the arguments of this lecture on a rigorous footing.