## 3 Full Fourier series

We saw in previous lectures how the Dirichlet and Neumann boundary conditions lead to respectively sine and cosine Fourier series of the initial data. In the same way the periodic boundary conditions

$$u(-l,t) = u(l,t), \quad u_x(-l,t) = u_x(l,t), \tag{1}$$

lead to the full Fourier expansion, i.e. the Fourier series that contains both sines and cosines. Such boundary conditions arise naturally in applications. For example heat conduction phenomena in a circular rod will be described by the heat equation subject to boundary conditions (1).

To see how the full Fourier series comes about, let us consider the eigenvalue problem associated with the boundary conditions (1),

$$\begin{cases} X'' = -\lambda X \\ X(-l) = X(l), \\ X'(-l) = X'(l). \end{cases} \tag{2}$$

To find the eigenvalues and eigenfunctions for (2), we consider the qualitatively distinct cases $\lambda < 0$, $\lambda = 0$ and $\lambda > 0$ separately.

We first assume $\lambda = -\gamma^2 < 0$. In this case the equation takes the form $X'' = \gamma^2 X$, and the solutions are thus $X(x) = Ce^{\gamma x} + De^{-\gamma x}$. The boundary conditions then imply

$$\begin{cases} Ce^{-\gamma l} + De^{\gamma l} = Ce^{\gamma l} + De^{-\gamma l}, \\ C\gamma e^{-\gamma l} - D\gamma e^{\gamma l} = C\gamma e^{\gamma l} - D\gamma e^{-\gamma l}. \end{cases}$$

Multiplying the first equation by $\gamma$ and adding to the second equation gives

$$2C\gamma e^{-\gamma l} = 2C\gamma e^{\gamma l} \quad \Rightarrow \quad 2C\gamma e^{\gamma l}(1 - e^{-2\gamma l}) = 0 \quad \Rightarrow \quad C = 0,$$

since $\gamma, l \neq 0$. Similarly, $D = 0$, which implies that there are no negative eigenvalues.

We next assume that $\lambda = 0$, which results in the equation $X'' = 0$. The solution is then $X(x) = C + Dx$, and the boundary conditions imply

$$\begin{cases} C - Dl = C + Dl \\ D = D \end{cases} \quad \Rightarrow \quad 2Dl = 0 \quad \Rightarrow \quad D = 0.$$

So $X(x) \equiv 1$ is an eigenfunction corresponding to the eigenvalue $\lambda_0 = 0$.

Finally, we consider the case $\lambda = \beta^2 > 0$. The equation then takes the form $X'' = -\beta^2 X$, and hence, has the solution $X(x) = C\cos\beta x + D\sin\beta x$. Checking the boundary conditions gives

$$\begin{cases} C\cos\beta l - D\sin\beta l = C\cos\beta l + D\sin\beta l \\ C\beta\sin\beta l + D\beta\cos\beta l = -C\beta\sin\beta l + D\beta\cos\beta l \end{cases} \quad \Rightarrow \quad \begin{cases} 2D\sin\beta l = 0 \\ 2C\beta\sin\beta l = 0 \end{cases}$$

Since $C$ and $D$ cannot both be equal to zero, and $\beta \neq 0$, we must have

$$\sin\beta l = 0 \quad \Rightarrow \quad \beta = \frac{n\pi}{l}, \quad \text{for } n = 1, 2, \dots.$$

Then the positive eigenvalues and the corresponding eigenfunctions are

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2, \quad X_n(x) = C\cos\frac{n\pi x}{l} + D\sin\frac{n\pi x}{l}, \quad \text{for } n = 1, 2, \dots.$$

Notice that for each of the eigenvalues $\lambda_n$, $n = 1, 2, \dots$ we have two linearly independent eigenfunctions, $\cos(n\pi x/l)$ and $\sin(n\pi x/l)$. This differs from the case of the Dirichlet and Neumann boundary conditions, where we had only one linearly independent eigenfunction for each of the same eigenvalues, namely $\sin(n\pi x/l)$ for Dirichlet, and $\cos(n\pi x/l)$ for Neumann. We will see in the next lecture that in the cases where there are more than one linearly independent eigenfunctions corresponding to the same eigenvalue, one can always choose a pairwise orthogonal set of eigenfunctions, which is necessary for the method of computing the Fourier coefficients to go through.

Having solved the eigenvalue problem (2), it is now clear that the series solutions of PDEs subject to the periodic boundary conditions (1) will have both sines and cosines in their expansions. For example the solution to the heat equation satisfying the boundary conditions (1) will be given by the series

$$u(x,t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} e^{-(n\pi/l)^2 kt}\left(A_n \cos\frac{n\pi x}{l} + B_n \sin\frac{n\pi x}{l}\right),$$

where the coefficients $A_n, B_n$ come from the expansion of the initial data

$$\phi(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{l} + B_n \sin\frac{n\pi x}{l}. \tag{3}$$

The series (3) is called the full Fourier series of the function $\phi(x)$ on the interval $(-l, l)$.

It is now clear that to solve boundary value problems with periodic boundary conditions (1) via separation of variables, one needs to find the coefficients in the expansion (3). This procedure is similar to finding the coefficients of the Fourier cosine and sine series, and relies on the pairwise orthogonality of the eigenfunctions. Notice that (3) is a linear combination of the elements of the set

$$\left\{1, \cos\frac{n\pi x}{l}, \sin\frac{n\pi x}{l} : n = 1, 2, \ldots\right\}. \tag{4}$$

Now, if we can prove that the elements of this set are pairwise orthogonal, then the coefficients in (3) can be found by taking the dot product of the series (3) with the corresponding eigenfunction. Since the interval in this case is $(-l, l)$, we define the dot product of two functions to be

$$(f, g) = \int_{-l}^{l} f(x)g(x)\,dx.$$

Let us now check the orthogonality of the set (4). We need to show that

$$\int_{-l}^{l} \cos\frac{n\pi x}{l} \sin\frac{m\pi x}{l}\,dx = 0, \quad \text{for all } n, m = 1, 2, \ldots, \tag{5}$$

$$\int_{-l}^{l} \cos\frac{n\pi x}{l} \cos\frac{m\pi x}{l}\,dx = 0, \quad \text{for } n \neq m, \tag{6}$$

$$\int_{-l}^{l} \sin\frac{n\pi x}{l} \sin\frac{m\pi x}{l}\,dx = 0, \quad \text{for } n \neq m, \tag{7}$$

$$\int_{-l}^{l} 1 \cdot \cos\frac{m\pi x}{l}\,dx = 0, \quad \text{for all } m = 1, 2, \ldots, \tag{8}$$

$$\int_{-l}^{l} 1 \cdot \sin\frac{m\pi x}{l}\,dx = 0, \quad \text{for all } m = 1, 2, \ldots. \tag{9}$$

Notice that, since $\sin(n\pi x/l)$ and $\cos(n\pi x/l)\sin(m\pi x/l)$ are odd functions, (5) and (9) are trivial, since the integration is performed over a symmetric interval. The integrands in (6), (7) and (8) are even functions, so the integrals are equal to twice the corresponding integrals over the interval $(0, l)$. Then, say for (6), we have

$$\int_{-l}^{l} \cos\frac{n\pi x}{l} \cos\frac{m\pi x}{l}\,dx = 2\int_{0}^{l} \cos\frac{n\pi x}{l} \cos\frac{m\pi x}{l}\,dx = 0, \quad \text{for } n \neq m,$$

as was computed in the last lecture, when considering Fourier cosine series over the interval $(0, l)$. Proofs of(7) and (8) are similar.

Thus, the set (4) is indeed orthogonal. To compute the coefficients in (3), we take the dot product of the series with the corresponding eigenfunctions. For example,

$$\int_{-l}^{l} \phi(x) \cos \frac{m\pi x}{l}\, dx = \int_{-l}^{l} \left( \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{l} + B_n \sin \frac{n\pi x}{l} \right) \cos \frac{m\pi x}{l} = A_m \int_{-l}^{l} \cos^2 \frac{m\pi x}{l}\, dx,$$

and similarly for the eigenfunctions $1 = \cos(0\pi x/l)$, and $\sin(m\pi x/l)$. But

$$\begin{aligned}
&\int_{-l}^{l} \cos^2 \frac{m\pi x}{l}\, dx = 2\int_{0}^{l} \cos^2 \frac{m\pi x}{l}\, dx = 2 \cdot \frac{l}{2} = l,\\
&\int_{-l}^{l} \sin^2 \frac{m\pi x}{l}\, dx = 2\int_{0}^{l} \sin^2 \frac{m\pi x}{l}\, dx = 2 \cdot \frac{l}{2} = l,\\
&\int_{-l}^{l} 1^2\, dx = 2l.
\end{aligned}$$

So the coefficients can be computed as

$$\begin{aligned}
A_m &= \frac{1}{l}\int_{-l}^{l} \phi(x) \cos \frac{m\pi x}{l}\, dx, \quad n = 0, 1, 2, \ldots,\\
B_m &= \frac{1}{l}\int_{-l}^{l} \phi(x) \sin \frac{m\pi x}{l}\, dx, \quad n = 1, 2, \ldots.
\end{aligned} \tag{10}$$

**Example 3.1.** Find the full Fourier series of the function $\phi(x) = x$ on the interval $(-l, l)$.

We compute the coefficients in the expansion

$$x = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{l} + B_n \sin \frac{n\pi x}{l}$$

using formulas (10).

$$A_n = \frac{1}{l}\int_{-l}^{l} x \cos \frac{n\pi x}{l}\, dx = 0,$$

since the function $x\cos(n\pi x/l)$ is odd, and the integration interval is symmetric. Using evenness of the function $x \sin(n\pi x/l)$, we have

$$B_n = \frac{1}{l}\int_{-l}^{l} x \sin \frac{n\pi x}{l}\, dx = \frac{2}{l}\int_{0}^{l} x \sin \frac{n\pi x}{l}\, dx = (-1)^{n+1}\frac{2l}{n\pi},$$

where the last integral is exactly the coefficient of the Fourier sine series of function $\phi(x) = x$, computed in the last lecture. Thus the full Fourier series of $x$ on the interval $(-l, l)$ is

$$x = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2l}{n\pi} \sin \frac{n\pi x}{l},$$

which exactly coincides with the Fourier sine series of the function $x$ on the interval $(0, l)$. □

### 3.1 Even, odd and periodic functions

In the previous example we could take any odd function $\phi(x)$, and the coefficients of the cosine terms in the full Fourier series would vanish for exactly the same reason, leading to the Fourier sine series. Thus the full Fourier series of an odd function on the interval $(-l, l)$ coincides with its Fourier sine series on the interval $(0, l)$. Analogously, the full Fourier series of an even function on the interval $(-l, l)$ coincides with its Fourier cosine series on the interval $(0, l)$. This is not surprising, taking into account the fact that the sine and cosine functions are themselves respectively odd and even.

So how can the same function $\phi(x)$, be it odd, even, or neither, have both a nonzero Fourier cosine and a nonzero Fourier sine expansion on the interval $(0, l)$? The answer to this question is trivial, if one recalls the reflection method used to solve boundary value problems on the finite interval $(0, l)$. In the reflection method one takes either an odd (Dirichlet BCs), or an even (Neumann BCs) extension of the data. Then the resulting extended data is either odd or even on the symmetric interval $(-l, l)$, and is extended to be periodic with period $2l$ to the rest of the number line. But then one can find the full Fourier series of this extended data, which will contain only sines or cosines, depending on the evenness or the oddness of the extension, thus giving either the Fourier cosine, or Fourier sine series of the function $\phi(x)$ on the interval $(0, l)$.

In short, one has the following correspondence between the boundary conditions, extensions, and Fourier series.

Dirichlet: $u(0, t) = u(l, t) = 0 \quad \to \quad$ odd extension $\quad \to \quad$ Fourier sine series.

Neumann: $u_x(0, t) = u_x(l, t) = 0 \quad \to \quad$ even extension $\quad \to \quad$ Fourier cosine series.

Periodic: $u(-l, t) = u(l, t), u_x(-l, t) = u_x(l, t) \quad \to \quad$ periodic extension $\quad \to \quad$ full Fourier series.

### 3.2 The complex form of the full Fourier series

Instead of writing the Fourier series in terms of cosines and sines, one can use complex exponential functions. Recall the Euler's formula,

$$e^{i\theta} = \cos\theta + i\sin\theta, \tag{11}$$

from which one can get the formula for $e^{-i\theta}$ in terms of cosine and sine by plugging in $-\theta$. Solving from these two equations, we have the expressions of sine and cosine in terms of the complex exponentials

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \quad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}. \tag{12}$$

So instead of (3), one can write the Fourier expansion in terms of the functions $e^{in\pi x/l}$ and $e^{-in\pi x/l}$, themselves eigenfunctions corresponding to the eigenvalue $\lambda_n = n^2\pi^2/l^2$.

The convenience of the complex form is in that the Fourier series can be written compactly as

$$\phi(x) = \sum_{n=-\infty}^{\infty} C_n e^{in\pi x/l}, \tag{13}$$

which is a linear combination in terms of the elements of the set

$$\left\{1, e^{i\pi x/l}, e^{-i\pi x/l}, e^{i2\pi x/l}, e^{-i2\pi x/l}, \dots\right\} = \left\{1, e^{\pm in\pi x/l} : n = 1, 2, \dots\right\}, \tag{14}$$

replacing the set (4).

In order to find the coefficients in the expansion (13), we first need to show that the set (14) is orthogonal, after which the coefficients can be found by taking the dot product of the series with the corresponding eigenfunction. Notice that the functions in the set (14) are complex valued. For complex valued functions the dot product is defined as

$$(f, g) = \int_{-l}^{l} f(x)\overline{g(x)}\,dx,$$

where the bar stands for the complex conjugate. Then the dot product of two distinct elements of the set (14) is

$$\int_{-l}^{l} e^{in\pi x/l} e^{-im\pi x/l}\, dx = \frac{1}{i(n-m)\pi} e^{i(n-m)\pi x/l}\bigg|_{-l}^{l} = \frac{e^{-i(n-m)\pi}}{i(n-m)\pi}(e^{i2(n-m)\pi} - 1) = 0, \quad \text{for } n \neq m.$$

So taking the dot product of the series (13) with one of the eigenfunctions in (14) gives

$$\int_{-l}^{l} \phi(x) e^{-im\pi x/l}\, dx = \int_{-l}^{l} \sum_{n=-\infty}^{\infty} C_n e^{in\pi x/l} e^{-im\pi x/l}\, dx = C_m \int_{-l}^{l} e^{im\pi x/l} e^{-im\pi x/l}\, dx = C_m \cdot 2l.$$

Thus, the coefficients can be computed as

$$C_n = \frac{1}{2l} \int_{-l}^{l} \phi(x) e^{-in\pi x/l}\, dx. \tag{15}$$

### 3.3 Conclusion

We saw how the separation of variables in the case of periodic boundary conditions leads to the full Fourier series (3) in the same way as the Dirichlet and Neumann boundary conditions lead to the Fourier sine and cosine series respectively. The coefficients in the full Fourier expansion were computed by making use of the pairwise orthogonality of eigenfunctions, resulting in the coefficients formulas (10). We also explored the connection between the Fourier cosine, respectively sine, expansions on the interval $(0, l)$ and the full Fourier expansion on the interval $(-l, l)$, using the idea of even and odd extensions. Finally using Euler's formula (11), we rewrote the full Fourier series in the complex form (13), the coefficients of which were again computed by taking the dot product of the series with the eigenfunctions, and relying on the fact that the distinct eigenfunctions are orthogonal. We will see in the next lecture that one always has orthogonal eigenfunctions, provided the boundary conditions are symmetric in an appropriate sense. This will lead to the notion of generalized Fourier series.