# A PRACTICAL SPLITTING METHOD FOR STIFF SDES WITH APPLICATIONS TO PROBLEMS WITH SMALL NOISE*

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**Abstract.** We present an easy to implement drift splitting numerical method for the approximation of stiff, nonlinear stochastic differential equations (SDEs). The method is an adaptation of the SBDF multi-step method for deterministic differential equations and allows for a semi-implicit discretization of the drift term to remove high order stability constraints associated with explicit methods. For problems with small noise, of amplitude $\epsilon$, we prove that the method converges strongly with order $O(\Delta t^2 + \epsilon\Delta t + \epsilon^2\Delta t^{1/2})$ and thus exhibits second order accuracy when the time step is chosen to be on the order of $\epsilon$ or larger. We document the performance of the scheme with numerical examples and also present as an application a discretization of the stochastic Cahn-Hilliard equation which removes the high order stability constraints for explicit methods.

**Key words.** stochastic differential equations, mean-square convergence, weak convergence, multi-step methods, IMEX methods, Cahn-Hilliard equation, conservative phase field models, Langevin equations.

**AMS subject classifications.** 60H35, 65L06

**1. Introduction.** Stochastic Partial Differential Equations (SPDEs) are an important and essential modeling tool in a wide range of fields from nonlinear filtering to continuum physics [1]. Often the SPDEs employed in modeling a physical process involve nonlinear and high order derivative terms and have an additional random force term, arising for example from Brownian motion. The Cahn-Hilliard equation with additive noise [2], [3], [4], [5] used as a model for phase separation in a binary alloy in the presence of thermal fluctuations (the noise term) illustrates well this type of SPDE:

$$\frac{d\phi}{dt} = -D\nabla^4\phi + \nabla^2\Psi'(\phi) + \eta, \tag{1.1}$$

where

$$\Psi'(\phi) = \phi^3 - a\phi, \tag{1.2}$$

and $\eta$ is gaussian with zero mean and correlation

$$\langle\eta(x,t)\eta(x',t')\rangle = -2\epsilon\delta(t-t')\nabla^2\delta(x-x'). \tag{1.3}$$

The mobility constant $D$ is related to the noise amplitude $\epsilon$ to comply with the fluctuation-dissipation theorem, i.e. $D \sim \epsilon$. For the interested reader, a further treatment of equation (1.1) is provided in Section 4.

The numerical integration of these SPDEs presents two main challenges: i) The stochastic nature of the equation makes the design of high order methods quite intricate and ii) the presence of high order derivatives leads to stiffness, in the sense that explicit methods require prohibitively small time steps. Methods which are easy to

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implement such as the Euler (also called Euler-Maruyama) method or a slight modification of it where the highest order linear term is handled implicitly will never attain a strong order of accuracy greater than 1/2 and will never attain a weak order of accuracy greater than 1. As far as stability is concerned, Euler's method applied to the deterministic Cahn-Hilliard equation, for example, has a fourth order time step restriction.

In this paper we devote our attention to solving a particular type of nonlinear SPDE where the drift term can be separated into a linear term containing high order derivatives and a remaining, smoother nonlinear term. The presence of high order derivatives imposes severe restrictions of the time steps allowed by explicit numerical schemes. We refer here to this problem as stiffness and call the corresponding SPDE stiff.

We take the approach of solving such equations using the the method of lines and therefore consider vector Itô stochastic differential equations of the form

$$X(t) = X(t_0) + \int_{t_0}^{t} f(X(s), s)ds + \int_{t_0}^{t} G(X(s), s)dW(s), \tag{1.4}$$

where the drift coefficient can be decomposed into the sum

$$f(x,t) = L(x,t) + N(x,t), \tag{1.5}$$

where $L(x,t)$, corresponding to high order derivatives, is linear and $N(x,t)$ is nonlinear. Here $W$ is an m-dimensional Wiener process corresponding to the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and the drift and diffusion functions are given as $f : \mathbb{R}^n \times \mathbb{R} \longrightarrow \mathbb{R}^n$ and $G = (g_1, ..., g_m) : \mathbb{R}^n \times \mathbb{R} \longrightarrow \mathbb{R}^{n \times m}$.

Motivated by the deterministic case, we propose a scheme for SDEs of the form (1.4)-(1.5) which is easy to implement, has low computational cost, and which exhibits improved stability (in the sense of absolute stability) over current methods. Furthermore, for problems with small noise the scheme exhibits second order accuracy (in time) when the time step is adequately selected.

The numerical scheme is a stochastic extension of the Extrapolated Gear multistep method [6] (also called Semi-implicit Backward Differentiation Formula or SBDF) and is given by:

$$\begin{aligned} 3X_\ell - 4X_{\ell-1} + X_{\ell-2} = \\ & 2\Delta t[L(X_\ell, t_\ell) + 2N(X_{\ell-1}, t_{\ell-1}) - N(X_{\ell-2}, t_{\ell-2})] \\ & +3G(X_{\ell-1}, t_{\ell-1})\Delta W(t_{\ell-1}) - G(X_{\ell-2}, t_{\ell-2})\Delta W(t_{\ell-2}) \end{aligned} \tag{1.6}$$

where $t_\ell$ is a uniform discretization of $\mathcal{T} = [t_0, t_N]$ and $\Delta W(t_{\ell-1}) = W(t_\ell) - W(t_{\ell-1})$. We refer here to this scheme as Stochastic SBDF or SSBDF.

The outline of the paper is as follows. In Section 2, we provide background information on stochastic multistep methods, SDEs with small noise, and absolute stability in the context of numerical methods for SDEs. In Section 3, we document the performance of the SSBDF scheme with numerical examples. In Section 4, we illustrate with an application the effectiveness of the SSBDF scheme at removing the stiffness associated with explicit methods applied to SPDEs. With a judicious choice of the drift coefficient splitting, we propose a discretization of the stochastic Cahn-Hilliard equation (1.1) which improves upon time step restrictions of current methods by several orders of magnitude. In Section 5, we state and prove a convergence theorem for the SSBDF scheme.

## 2. Background.

**2.1. Multistep methods and their application to SDEs with small noise.** The SSBDF scheme (1.6) is an example of a stochastic multistep method; a numerical method which uses past values of the numerical iterate and past values of the Wiener process to approximate future values of the numerical iterate. In general, as pointed out in [7], since the Wiener process and the solution of (1.4) are not differentiable, high order convergence is only possible when the multi-step scheme includes sufficient information about the Wiener process (through multiple stochastic integrals). Multistep methods which only use Wiener increments as in (1.6) will only attain strong convergence of order 1/2. But convergence is an asymptotic property and when the influence of noise is not dominant it is possible to retain some of the behavior of the multistep methods in the deterministic setting for suitable step sizes [7].

For example, consider the SDE (1.4) with small noise coefficient $G(x,t) = \epsilon\hat{G}(x,t)$:

$$X(t) = X(t_0) + \int_{t_0}^{t} f(X(s),s)ds + \epsilon \int_{t_0}^{t} \hat{G}(X(s),s)dW(s), \tag{2.1}$$

where $f(x,t) = L(x,t) + N(x,t)$ and $\hat{G}(x,t)$ are $O(1)$ and $\epsilon$ is a small parameter. We prove in Section 5 that, given certain smoothness and growth conditions on the drift and diffusion coefficients, the global strong error of the SSBDF scheme applied to the SDE (2.1) with small noise is of order $O(\epsilon^2\Delta t^{1/2} + \epsilon\Delta t + \Delta t^2)$. Thus when $\Delta t \geq O(\epsilon)$, the SSBDF scheme exhibits second order accuracy.

When the method of lines is applied to SPDEs, the resulting system of SDEs is often of the form

$$X(t) = X(t_0) + \lambda_1 \int_{t_0}^{t} f(X(s),s)ds + \lambda_2 \int_{t_0}^{t} \hat{G}(X(s),s)dW(s), \tag{2.2}$$

where $f$ and $\hat{G}$ are $O(1)$ and $|\lambda_1| >> |\lambda_2|$ due to the presence of high order derivatives. With a time rescaling this system can be recast into the form (2.1).

We note that numerical methods already appear in the literature for SDEs with small noise. In [8], [9] Milstein and Tretyakov derive explicit and implicit Taylor and Runga-Kutta methods for SDEs with small noise and in [7] Buckwar and Winkler derive explicit and implicit multistep methods for SDEs with small noise. Additionally in [7], the general theory of the mean-square (strong) convergence of multistep methods applied to stochastic differential equations is developed. The analysis mirrors that of the theory for deterministic multistep methods and stochastic analogs of zero stability and consistency are defined to prove convergence.

**2.2. Stiff SDEs and absolute stability.** When an explicit method is applied to a stiff differential equation, the time step size necessary for the numerical iterate to remain bounded is much smaller than the time step size necessary to resolve accurately the underlying solution. The time steps for which the iterate remains bounded determine the *region of absolute stability.*

To gain insight into how large time steps can be taken in practice for general differential equations, it is useful to determine the region of absolute stability of a numerical scheme applied to a test equation. In [10], the complex test equation

$$dX_t = \alpha X_t + dW_t \tag{2.3}$$

with $Re(\alpha) < 0$ is considered. The region of absolute stability of a one-step method applied to (2.3) of the form

$$X_\ell = G(\alpha\Delta t) \cdot X_{\ell-1} + Z_{\ell-1}, \tag{2.4}$$

where $Z_\ell$ are random variables independent of $\alpha$ and $X_\ell$, is the set of complex numbers $\alpha\Delta t$ such that given two arbitrary deterministic initial conditions $X_0$ and $Y_0$ ($X_0 \neq Y_0$) the inequality

$$|\xi_\ell| = |X_\ell - Y_\ell| < |\xi_{\ell-1}| \tag{2.5}$$

holds for all trajectories. Equivalently, the region of absolute stability is the set of complex numbers $\alpha\Delta t$ such that $|G(\alpha\Delta t)| < 1$.

We point out, as is done in [10], that for methods of the form (2.4) applied to (2.3), the region of absolute stability is the same as the region of absolute stability in the deterministic case when (2.3) has no additive noise. This follows because the discrete noise terms $Z_\ell$ cancel out in (2.5) and the $\xi_\ell$ satisfy the deterministic recursion formula $\xi_\ell = G(\alpha\Delta t) \cdot \xi_{\ell-1}$.

For splitting schemes like the SSBDF scheme (1.6), we rewrite the test equation (2.3) in the form:

$$dX_t = \alpha X_t + i\beta X_t + dW_t, \tag{2.6}$$

where $\alpha < 0$ and $\beta$ are real. Equation (2.6) with zero noise is considered in [11] where it corresponds to a Fourier mode of the linear advection-diffusion equation

$$u_t = a u_{xx} + b u_x. \tag{2.7}$$

The regions of absolute stability of several deterministic splitting schemes, including SBDF, applied to (2.6) with zero noise are compared in [11]. Since the first term in (2.6) corresponds to the second derivative in the advection-diffusion equation and the second term corresponds to the first derivative we have that $|\alpha| \gg |\beta|$ for high frequency modes. Thus the first term in (2.6) should be treated implicitly by splitting schemes and the second term explicitly.

To find the region of absolute stability of the SSBDF scheme applied to (2.6) we note that given deterministic initial conditions $X_0$, $X_1$, $Y_0$, and $Y_1$ the error $\xi_\ell = X_\ell - Y_\ell$ solves the recursion formula

$$3\xi_\ell - 4\xi_{\ell-1} + \xi_{\ell-2} = 2\Delta t(\alpha\xi_\ell + 2\beta i\xi_{\ell-1} - \beta i\xi_{\ell-2}) \tag{2.8}$$

since the noise terms cancel out. Thus the region of absolute stability is the same as in the deterministic case and can be found via the characteristic polynomial of the deterministic scheme.

When $|\alpha| >> |\beta|$, it turns out that the SSBDF scheme has a very large region of absolute stability and thus a mild time step restriction when applied to the test equation (2.6) (see Figure 2.1). For this reason, we find that the SSBDF scheme performs well when applied to SDEs which allow the splitting (1.5). In Example 3.2 in Section 3 and Section 4 on the Cahn-Hilliard equation we find that, for the chosen parameter values, the SSBDF scheme allows for time steps several orders of magnitude greater than the time steps allowed by explicit methods.

We note here that other splitting schemes have been proposed in the literature for the approximation of stiff SDEs. In [12], Petersen analyzes a second order accurate scheme for the weak approximation of SDEs where the drift term is discretized by

$$f(X_{\ell+1/2}) \approx \frac{1}{2}[L(X_{\ell+1}) + L(X_\ell) + N(X_{Euler}) + N(X_\ell)]. \tag{2.9}$$

The discretization (2.10) is similar to a Crank-Nicolson discretization except for the $X_{Euler}$ term which corresponds to a prediction step using the Forward Euler scheme. This scheme has a mild time step restriction when applied to (2.6) for $|\beta| = O(1)$, however, when $|\beta| >> 1$ the $X_{Euler}$ term leads to a small region of absolute stability and a stringent time step restriction of the form $|\Delta t| \leq \frac{C}{|\beta|}$ (see Figure 2.1).

Another splitting scheme, useful for initializing the SSBDF scheme, is the Semi-Implicit (SI), or modified, Euler scheme:

$$X_\ell - X_{\ell-1} = \Delta t[L(X_\ell, t_\ell) + N(X_{\ell-1}, t_{\ell-1})] + G(X_{\ell-1}, t_{\ell-1})\Delta W(t_{\ell-1}). \tag{2.10}$$

Along with the SSBDF scheme, SI Euler has a large region of absolute stability when applied to (2.6). When applied to the SDE (2.1) with small noise coefficient $G = \epsilon\hat{G}$, we find with a similar estimate as in Section 5 that the global strong order of convergence of the SI Euler scheme is $O(\epsilon^2\Delta t^{1/2} + \Delta t)$. However, the local error is order $O(\epsilon^2\Delta t + \epsilon\Delta t^{3/2} + \Delta t^2)$ and thus the SI Euler scheme can be used to initialize the SSBDF scheme without degrading the overall accuracy or stability.

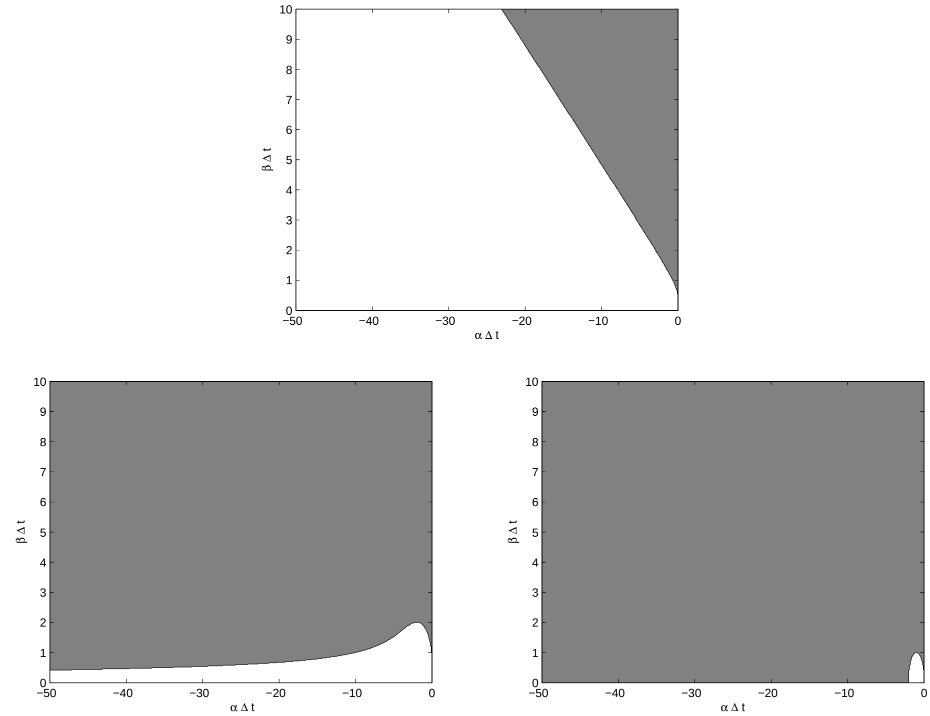


FIG. 2.1. *Regions of absolute stability (white) for the SSBDF scheme (top), Petersen's scheme (left), and Forward Euler (right) applied to the test equation (2.6).*

## 3. Numerical Experiments.

**3.1. Example 1.** We first apply the SSBDF scheme to the linear test equation

$$dX_t = -X_t \cdot dt + \epsilon X_t \cdot dW_t,\ \ X(0) = 1, \tag{3.1}$$

where the exact solution is given by

$$X(t) = \exp\left(\left(-1 - \frac{1}{2}\epsilon^2\right)t + \epsilon W(t)\right). \tag{3.2}$$

For this example we choose the splitting $L(x,t) = -0.9x$ and $N(x,t) = -0.1x$. For all of the examples considered in this paper the SSBDF scheme is initialized with one step of the SI Euler scheme.

We compute approximations of both the strong error and the weak error of the numerical scheme. The strong error is approximated for the numerical solution of (3.1) at time $t_N = 1$ by

$$\|X(t_N) - X_N\|_{L_2} \approx \left(\frac{1}{M}\sum_{j=1}^{M} |X(t_N,\omega_j) - X_N(\omega_j)|^2\right)^{\frac{1}{2}} \tag{3.3}$$

along with the Standard Error (S.E.) associated with this Monte-Carlo approximation. Here the Standard Error of a sample $\{Y_i\}_{i=1}^N$ refers to the statistic

$$S.E. = \sqrt{\frac{\sum_{i=1}^{N}(Y_i - \overline{Y})^2}{N(N-1)}}, \tag{3.4}$$

where $\overline{Y}$ is the sample mean.

As is done in [10], we use the term weak convergence synonymously with convergence of moments. We thus illustrate weak convergence by approximating the pth moment of the numerical solution of (3.1) at time $t_N = 1$ by

$$\Big|E(X(t_N))^p - E(X_N)^p\Big| \approx \Big|E(X(t_N))^p - \frac{1}{M}\sum_{j=1}^{M} X_N(\omega_j)^p\Big| \tag{3.5}$$

along with the Standard Error associated with this Monte-Carlo approximation.

In Tables 3.1-3.3 we document the strong convergence of the SSBDF scheme for decreasing values of the noise coefficient $\epsilon$. In Table 3.1 ($\epsilon = .1$) we observe a decrease in the order of convergence towards 1/2 as the time step size is decreased. As the noise is decreased in Tables 3.2 and 3.3, we see improvement in the order of convergence towards the deterministic order of convergence 2. In Table 3.4 we observe that the SSBDF scheme has a higher order of convergence in the weak sense, for the test equation (3.1). Here the noise is the same as in Table 3.1 ($\epsilon = .1$), but the order of convergence stays above 1 as the time step size is decreased. This is not surprising since schemes of strong order 1/2 are often order 1 weakly [10].

**3.2. Example 2.** Next we apply the SSBDF scheme to the stiff SDE

$$dX_t = -10^8(X_t - \sin(5t)) \cdot dt + 10^6 \sin(X_t) \cdot dt + 10^4 \cdot dW_t,\ \ X(0) = 0. \tag{3.6}$$

Note the similarities with equation (1.1). When equation (1.1) is discretized in space with $D = \epsilon = 10^{-4}$ and $\Delta x = 10^{-3}$, the resulting system of stochastic ODEs contains

TABLE 3.1
*Example 1: Strong Error, $\epsilon = .1$*

| $\Delta t$ | *Error* $\pm$ *S.E.* | *Order* |
|---|---|---|
| .10000 | .00263 ± .00011 | |
| .05000 | .00119 ± .00005 | 1.14 |
| .02500 | .00064 ± .00003 | 0.89 |
| .01250 | .00038 ± .00002 | 0.75 |
| .00625 | .00024 ± .00002 | 0.66 |

TABLE 3.2
*Example 1: Strong Error, $\epsilon = .01$*

| $\Delta t$ | *Error* $\pm$ *S.E.* | *Order* |
|---|---|---|
| .10000 | .001464 ± .000051 | |
| .05000 | .000361 ± .000018 | 2.02 |
| .02500 | .000098 ± .000006 | 1.88 |
| .01250 | .000032 ± .000002 | 1.61 |
| .00625 | .000013 ± .000001 | 1.30 |

a linear term with negative eigenvalues on the order of $10^8$, a nonlinear term of order $10^6$, and noise with a coefficient on the order of $10^4$.

Referring back to Section 2.2, standard explicit methods have a time step restriction of the form $\Delta t \leq \frac{C}{10^8}$ when applied to (3.6) and are thus impractical. The scheme analyzed in [12], that uses the drift splitting (2.9), removes the stiffness contributed from the leading order term but still has a time step restriction of the form $\Delta t \leq \frac{C}{10^6}$. Fully implicit methods require the use of a nonlinear solver such as Newton's method, which in higher dimensions can be challenging to implement and computationally expensive.

The SSBDF scheme, on the other hand, removes the stiffness contributed by both the linear and nonlinear term and is straightforward to implement. We set $L(x,t) = -10^8(x - \sin(5t))$, $N(x,t) = 10^6 \sin(x)$ and integrate up to time $t_N = 1$. We calculate $E[F(X_N)]$, where $F(x) = -x + x^2 - x^3$, using the Monte-Carlo approximation (3.5) along with the Standard Error of the Monte-Carlo approximation. Here $F(x)$ is chosen such that the first, second, and third moments are represented equally ($X_N \approx -1$).

Letting $X^{\Delta t}$ denote the approximation at time $t = 1$ corresponding to the time step size $\Delta t$, we estimate the error of the approximation and the order of convergence by

$$error_{\Delta t} \approx \left| E[F(X^{2\Delta t})] - E[F(X^{\Delta t})] \right| \tag{3.7}$$

and

$$order_{\Delta t} \approx \log(error_{2\Delta t}/error_{\Delta t})/\log(2). \tag{3.8}$$

In Table 3.5 we document the weak convergence of the SSBDF scheme applied to (3.6) with the chosen function $F(x)$ above. Here the time step size is chosen to resolve the slow time scale associated with the function $\sin(5t)$ and we observe in Table 3.5 second order convergence. We note again that the time steps used in Table 3.5 are

TABLE 3.3
*Example 1: Strong Error, $\epsilon = .001$*

| $\Delta t$ | *Error* $\pm$ *S.E.* | *Order* |
|---|---|---|
| .10000 | .0014501 $\pm$ .0000162 | |
| .05000 | .0003481 $\pm$ .0000055 | 2.06 |
| .02500 | .0000856 $\pm$ .0000019 | 2.02 |
| .01250 | .0000213 $\pm$ .0000007 | 2.01 |
| .00625 | .0000054 $\pm$ .0000002 | 1.98 |

TABLE 3.4
*Example 1: Weak Error, $\epsilon = .1$*

| *Moment* | $\Delta t$ | *Error* $\pm$ *S.E.* | *Order* |
|---|---|---|---|
| First Moment | .10000 | .0014500 $\pm$ .0000146 | |
| | .05000 | .0003479 $\pm$ .0000039 | 2.06 |
| | .02500 | .0000855 $\pm$ .0000019 | 2.02 |
| | .01250 | .0000212 $\pm$ .0000017 | 2.01 |
| | .00625 | .0000053 $\pm$ .0000017 | 2.00 |
| Second Moment | .10000 | .0012164 $\pm$ .0000122 | |
| | .05000 | .0003271 $\pm$ .0000035 | 1.89 |
| | .02500 | .0000977 $\pm$ .0000016 | 1.74 |
| | .01250 | .0000328 $\pm$ .0000013 | 1.57 |
| | .00625 | .0000124 $\pm$ .0000013 | 1.40 |
| Third Moment | .10000 | .0007618 $\pm$ .0000077 | |
| | .05000 | .0002219 $\pm$ .0000023 | 1.78 |
| | .02500 | .0000739 $\pm$ .0000010 | 1.59 |
| | .01250 | .0000279 $\pm$ .0000008 | 1.41 |
| | .00625 | .0000117 $\pm$ .0000007 | 1.25 |
| Fourth Moment | .10000 | .0004234 $\pm$ .0000043 | |
| | .05000 | .0001309 $\pm$ .0000014 | 1.69 |
| | .02500 | .0000467 $\pm$ .0000006 | 1.49 |
| | .01250 | .0000188 $\pm$ .0000004 | 1.31 |
| | .00625 | .0000082 $\pm$ .0000004 | 1.20 |

outside of the regions of absolute stability of explicit schemes and the scheme with splitting (2.9).

TABLE 3.5
*Example 2: Weak Error*

| *Method* | $\Delta t$ | $E[F(X_N)] \pm S.E.$ | *Order* |
|---|---|---|---|
| SSBDF | .10000 | 2.81607 $\pm$ .00001 | |
| | .05000 | 2.80919 $\pm$ .00001 | |
| | .02500 | 2.80773 $\pm$ .00001 | 2.23 |
| | .01250 | 2.80736 $\pm$ .00001 | 1.98 |
| | .00625 | 2.80728 $\pm$ .00001 | 2.21 |

**4. The stochastic Cahn-Hilliard equation.** In this section we illustrate with the stochastic Cahn-Hilliard equation (1.1) how the SSBDF scheme can be implemented for stochastic partial differential equations to effectively remove the stiffness associated with explicit methods.

Equation (1.1) has been investigated numerically in [3], [4], [5] to study its appropriateness for modeling phase separation. Following [3], we apply the method of lines to (1.1) with periodic boundary conditions on the unit square, replacing the Laplacian $\nabla^2$ by the standard discrete Laplacian $\mathcal{L}$:

$$\mathcal{L}\phi_{i,j}(t) = \frac{1}{(\Delta x)^2}\big(\phi_{i+1,j}(t) + \phi_{i-1,j}(t) + \phi_{i,j+1}(t) + \phi_{i,j-1}(t) - 4\phi_{i,j}(t)\big). \tag{4.1}$$

Here $(i,j)$ denotes a grid point in the spatial mesh. We discretize the noise term $\eta$, as was done in [3], by substituting the Kroneker delta $\delta_{i,j}$ and discrete Laplacian into (1.3) and get

$$\begin{aligned}\langle \eta_{i_1,j_1}(t)\eta_{i_2,j_2}(t')\rangle =& \frac{-2\epsilon\delta(t-t')}{(\Delta x)^4}(-4\delta_{i_1-i_2,j_1-j_2} + \delta_{i_1-i_2+1,j_1-j_2} \\ &+\delta_{i_1-i_2-1,j_1-j_2} + \delta_{i_1-i_2,j_1-j_2+1} + \delta_{i_1-i_2,j_1-j_2-1}).\end{aligned} \tag{4.2}$$

Thus when $(i_1, j_1)$ and $(i_2, j_2)$ are equal

$$\langle \eta_{i_1,j_1}(t)\eta_{i_2,j_2}(t')\rangle = \frac{8\epsilon\delta(t-t')}{(\Delta x)^4}$$

and when $(i_1, j_1)$ and $(i_2, j_2)$ are nearest neighbors

$$\langle \eta_{i_1,j_1}(t)\eta_{i_2,j_2}(t')\rangle = \frac{-2\epsilon\delta(t-t')}{(\Delta x)^4}.$$

Otherwise the correlation is zero. We can achieve these correlations by setting

$$\begin{aligned}\int_0^t \eta_{i,j}(s)ds = \left[\frac{2\epsilon}{(\Delta x)^4}\right]^{\frac{1}{2}} \cdot \Big(&\int_0^t dW^1_{i+1,j}(s) - \int_0^t dW^1_{i,j}(s) \\ &+\int_0^t dW^2_{i,j+1}(s) - \int_0^t dW^2_{i,j}(s)\Big),\end{aligned} \tag{4.3}$$

where $W^1_{i,j}(s)$, $W^2_{i,j}(s)$ are independent standard Brownian motions. By combining (4.1) and (4.3) we transform (1.1) into the form of (1.4):

$$\phi_{i,j}(t) = \phi_{i,j}(0) + \int_0^t \big[-D\mathcal{L}^2\phi_{i,j}(s) + \mathcal{L}\Psi'(\phi_{i,j}(s))\big]ds + \int_0^t \eta_{i,j}(s)ds. \tag{4.4}$$

To choose the splitting of the drift term, we observe that given a deterministic diffusion equation of the form

$$\frac{d\phi}{dt} = \nabla\cdot(a(\phi)\nabla\phi), \tag{4.5}$$

the finite difference scheme

$$\frac{\phi(t+\Delta t) - \phi(t)}{\Delta t} = \alpha\nabla^2\phi(t+\Delta t) + \nabla\cdot(a(\phi(t))\nabla\phi(t)) - \alpha\nabla^2\phi(t) \tag{4.6}$$

is unconditionally stable when $\alpha \geq \frac{1}{2}\max|a(\phi)|$ [13].

As is done in [14], we can apply the same idea to the deterministic Cahn-Hilliard equation for we have that

$$\nabla^2 \Psi'(\phi) = \nabla \cdot (\Psi'' \nabla \phi). \tag{4.7}$$

Using the bound

$$|\phi| \leq \sqrt{a} \tag{4.8}$$

which exists on the solution (where $a$ is as defined in (1.2)), we set

$$\alpha = |\Psi''(\sqrt{a})|, \tag{4.9}$$

$$L(\phi) = -D\nabla^4 \phi + \alpha \nabla^2 \phi, \tag{4.10}$$

and

$$N(\phi) = \nabla^2 \Psi'(\phi) - \alpha \nabla^2 \phi. \tag{4.11}$$

When we do so we find via numerical experiments that the SBDF scheme exhibits no time step restriction.

For the Cahn-Hilliard equation with additive noise we proceed similarly. Since the bound (4.8) no longer holds in the stochastic case we fix $\alpha$ to be the corresponding $\alpha$ from the deterministic case. We then use the splitting (4.10) and (4.11) for the drift term. To initialize the SSBDF scheme we use one step of SI Euler (using the splitting (4.10) and (4.11) and the same $\alpha$.)

When we do so we observe that for large time step sizes and long times of integration a percentage (which decreases as the time step size decreases) of the numerical trajectories explode in finite time. In [15], Milstein and Tretyakov address this issue and show that numerical methods which produce exploding trajectories can still be used for the approximation of SDEs by rejecting the trajectories which leave a sufficiently large sphere. We thus can use the SSBDF scheme, even with very large step sizes, applying this criterion. But for moderate size time steps we find that the SSBDF scheme produces no exploding trajectories .

For instance, we solve (4.4) (using Fast Fourier Transforms) on the square $[0,1] \times [0,1]$ with periodic boundary conditions and random initial conditions $\phi_{i,j}(0) = D \cdot U_{i,j}(-1,1)$ where the $U_{i,j}(-1,1)$ are independent uniform random variables. With $a = 2$, $\alpha = 5$, $D = 2\epsilon = 10^{-4}$ and $\Delta x = .1 \cdot \Delta t = .002$ we observed no exploding trajectories when 100 trajectories of (4.4) were simulated up to time $t = 1$ (see Table 4.1).

Table 4.1
*SSBDF with parameters $a = 2$, $\alpha = 5$, $D = 2\epsilon = 10^{-4}$*

| $t_{final}$ | $\Delta t$ | $\Delta x$ | *number of exploding trajectories* |
|---|---|---|---|
| 10 | .2 | .002 | 28/100 |
| 1 | .02 | .002 | 0/100 |

In comparison, the Forward Euler scheme (used in [4], [5]) has a fourth order time step restriction and thus produces exploding trajectories when time steps as small as $\Delta t = 2 \cdot 10^{-8}$ are used, as demonstrated in Table 4.2

TABLE 4.2
*Forward Euler with parameters $a = 2$ and $D = 2\epsilon = 10^{-4}$*

| $t_{final}$ | $\Delta t$ | $\Delta x$ | *number of exploding trajectories* |
|---|---|---|---|
| $10^{-6}$ | $2 \cdot 10^{-8}$ | .002 | 100/100 |
| $10^{-7}$ | $2 \cdot 10^{-9}$ | .002 | 0/100 |

We note that nonlinear schemes combined with a Crank-Nicolson type discretization have also been used to simulate both the deterministic Cahn-Hilliard equation [16] and the stochastic Cahn-Hilliard equation [3]. These methods can yield fruitful results, however, special care must be taken in choosing the iterative solver for the nonlinear system of equations at each time step to avoid computational expense and numerical instabilities (see [16] for more details).

**5. Convergence Proof.** In this section we state and prove a theorem concerning the strong convergence of the SSBDF scheme (1.6) applied to SDEs of the form (2.1) with small noise. We follow the work of Buckwar and Winkler in [7] on the strong convergence of general two step schemes applied to SDEs with small noise. In [7], consistency and numerical stability of general multistep methods are defined and then shown to be sufficient for strong convergence. Numerical stability follows from the multistep scheme satisfying Dahlquist's root condition and the coefficients of the SDE (2.1) satisfying Lipschitz conditions. Consistency is proved, provided that the coefficients of the SDE are sufficiently smooth and satisfy a linear growth bound, by first applying Ito's Lemma to expand the local error formula and then applying Lemma 2.12 in [7] to obtain local order estimates.

As is done in [7], we denote a multiple Wiener integral by

$$I^{t,t+\Delta t}_{r_1,...,r_j} = \int_t^{t+\Delta t} \int_t^{s_1} ... \int_t^{s_{j-1}} dW_{r_1}(s_j)...dW_{r_j}(s_1), \tag{5.1}$$

where $r_i \in \{0, ..., m\}$, $dW_0(s) = ds$, and a general multiple Wiener integral by

$$I^{t,t+\Delta t}_{r_1,...,r_j}(F) = \int_t^{t+\Delta t} \int_t^{s_1} ... \int_t^{s_{j-1}} F(X(s_j), s_j)dW_{r_1}(s_j)...dW_{r_j}(s_1). \tag{5.2}$$

We say that a function $F(x,t)$ satisfies a Lipschitz condition uniformly in $x$ if

$$|F(x,t) - F(y,t)| \leq L|x - y| \quad \forall x, y \in \mathbb{R}^n, t \in \mathcal{T}, \tag{5.3}$$

and satisfies a linear growth bound uniformly in $x$ if

$$|F(x,t)| \leq K(1 + |x|^2)^{1/2} \quad \forall x \in \mathbb{R}^n, t \in \mathcal{T}. \tag{5.4}$$

We also make use of the following operators which are defined in [7] to simplify expansions of the local error:

$$\Lambda_0 F = F'_t + F'_x f + \frac{1}{2}\sum_{r=1}^{m} F''_{xx}[g_r, g_r],$$

$$\Lambda_r F = F'_x g_r,$$

$$\Lambda_0^f F = F_t' + F_x' f,$$

$$\hat{\Lambda}_0 F = \frac{1}{2}\sum_{r=1}^{m} F_{xx}''[\hat{g}_r, \hat{g}_r],$$

$$\hat{\Lambda}_r F = F_x' \hat{g}_r.$$

With the above notations and definitions we now state the convergence theorem.

THEOREM 5.1. *Suppose that the initial value $X(t_0)$ of the SDE (2.1) is $L_2$ integrable and*
*I. The coefficients $f$ and $g_r = \epsilon \hat{g}_r$ of the SDE (2.1) with small noise are continuous and satisfy Lipschitz conditions of the form (5.3),*
*II. $f$, $g_r$ and $\Lambda_0^f f$ have continuous partial derivatives up to order 2 in $x$ and 1 in $t$,*
*III. $\Lambda_0 f$, $\Lambda_0 g_r$, $\hat{\Lambda}_r f$, $\hat{\Lambda}_q g_r$, $\Lambda_0 \Lambda_0^f f$, and $\hat{\Lambda}_r \Lambda_0^f f$ satisfy a linear growth bound of the form (5.4).*

*Then the global error of the SSBDF scheme (1.6) applied to the SDE (2.1) satisfies*

(5.5)
$$\max_{\ell=2,\dots,N} \|X(t_\ell) - X_\ell\|_{L_2} = O(\epsilon^2 \Delta t^{1/2} + \epsilon \Delta t + \Delta t^2) + O(\max_{\ell=0,1} \|X(t_\ell) - X_\ell\|_{L_2}).$$

*Proof of Theorem 5.1.*

First note that $L$ and $N$ in (1.6) and their derivatives satisfy conditions I-III since $L$ is linear and $N = f - L$. The numerical stability of the SSBDF scheme in the sense of Definition 2.6 in [7] then follows from $L$ and $N$ satisfying condition $I$ (the Lipschitz condition) and the fact that the characteristic polynomial of the SSBDF scheme satisfies Dahlquist's root condition. The proof is the same as the proof of Theorem 3.2 in [7].

The order estimate (5.5) then follows from equation (2.8) in [7], by setting $D_\ell = L_\ell$, if the local error of the SSBDF scheme allows the expansion $L_\ell = R_\ell^\circ + S_{1,\ell}^\circ + S_{2,\ell-1}^\circ$ where $E(S_{1,\ell}^\circ | \mathcal{F}_{t_{\ell-1}}) = 0$ and $E(S_{2,\ell-1}^\circ | \mathcal{F}_{t_{\ell-2}}) = 0$ and

$$\|R_\ell^\circ\|_{L_2} = O(\Delta t^3 + \epsilon^2 \Delta t^2), \tag{5.6}$$

$$\|S_{1,\ell}^\circ\|_{L_2} = O(\epsilon^2 \Delta t + \epsilon \Delta t^{3/2}), \tag{5.7}$$

and

$$\|S_{2,\ell-1}^\circ\|_{L_2} = O(\epsilon^2 \Delta t + \epsilon \Delta t^{3/2}). \tag{5.8}$$

Thus we just need to obtain the estimates (5.6), (5.7) and (5.8) and show that the random variables in (5.7) and (5.8) have zero expectation.

The local error of the SSBDF applied to (2.1) as defined by Definition 2.4 in [7] is given by

$$\begin{aligned} L_\ell = {} & (X(t_\ell) - X(t_{\ell-1})) - \frac{1}{3}(X(t_{\ell-1}) - X(t_{\ell-2})) \\ & - \frac{2\Delta t}{3}(L(X(t_\ell), t_\ell) + 2N(X(t_{\ell-1}), t_{\ell-1}) - N(X(t_{\ell-2}), t_{\ell-2})) \\ & - \sum_{r=1}^{m} g_r(X(t_{\ell-1}), t_{\ell-1}) I_r^{t_{\ell-1}, t_\ell} + \frac{1}{3}\sum_{r=1}^{m} g_r(X(t_{\ell-2}), t_{\ell-2}) I_r^{t_{\ell-2}, t_{\ell-1}}. \end{aligned} \tag{5.9}$$

From Itô's lemma we have that

(5.10)

$$N(X(t_{\ell-1}),t_{\ell-1}) = N(X(t_{\ell-2}),t_{\ell-2}) + I_0^{t_{\ell-2},t_{\ell-1}}(\Lambda_0 N) + \sum_{r=1}^{m} I_r^{t_{\ell-2},t_{\ell-1}}(\Lambda_r N)$$

and

(5.11)

$$\begin{aligned} L(X(t_\ell),t_\ell) &= L(X(t_{\ell-2}),t_{\ell-2}) + I_0^{t_{\ell-2},t_{\ell-1}}(\Lambda_0 L) + I_0^{t_{\ell-1},t_\ell}(\Lambda_0 L) \\ &+ \sum_{r=1}^{m} I_r^{t_{\ell-2},t_{\ell-1}}(\Lambda_r L) + \sum_{r=1}^{m} I_r^{t_{\ell-1},t_\ell}(\Lambda_r L). \end{aligned}$$

The SDE (2.1) implies the two identities

(5.12)

$$\begin{aligned} X(t_{\ell-1}) - X(t_{\ell-2}) &= \Delta t f(X(t_{\ell-2}),t_{\ell-2}) + I_{00}^{t_{\ell-2},t_{\ell-1}}(\Lambda_0 f) \\ &+ \sum_{r=1}^{m} I_{r0}^{t_{\ell-2},t_{\ell-1}}(\Lambda_r f) + \sum_{r=1}^{m} g_r(X(t_{\ell-2}),t_{\ell-2}) I_r^{t_{\ell-2},t_{\ell-1}} \\ &+ \sum_{r=1}^{m} I_{0r}^{t_{\ell-2},t_{\ell-1}}(\Lambda_0 g_r) + \sum_{r,q=1}^{m} I_{qr}^{t_{\ell-2},t_{\ell-1}}(\Lambda_q g_r) \end{aligned}$$

and, additionally using Itô's lemma,

(5.13)

$$\begin{aligned} X(t_\ell) - X(t_{\ell-1}) &= \Delta t\{f(X(t_{\ell-2}),t_{\ell-2}) + I_0^{t_{\ell-2},t_{\ell-1}}(\Lambda_0 f) + \sum_{r=1}^{m} I_r^{t_{\ell-2},t_{\ell-1}}(\Lambda_r f)\} \\ &+ I_{00}^{t_{\ell-1},t_\ell}(\Lambda_0 f) + \sum_{r=1}^{m} I_{r0}^{t_{\ell-1},t_\ell}(\Lambda_r f) + \sum_{r=1}^{m} g_r(X(t_{\ell-1}),t_{\ell-1}) I_r^{t_{\ell-1},t_\ell} \\ &+ \sum_{r=1}^{m} I_{0r}^{t_{\ell-1},t_\ell}(\Lambda_0 g_r) + \sum_{r,q=1}^{m} I_{qr}^{t_{\ell-1},t_\ell}(\Lambda_q g_r). \end{aligned}$$

Inserting these identities into the local error formula yields

$$L_\ell = R_\ell + S_{1,\ell} + S_{2,\ell-1} \tag{5.14}$$

where

$$\begin{aligned} R_\ell &= \Delta t I_0^{t_{\ell-2},t_{\ell-1}}(\Lambda_0 f) + I_{00}^{t_{\ell-1},t_\ell}(\Lambda_0 f) - \frac{1}{3} I_{00}^{t_{\ell-2},t_{\ell-1}}(\Lambda_0 f) \\ &- \frac{2\Delta t}{3}\{I_0^{t_{\ell-2},t_{\ell-1}}(\Lambda_0 L) + I_0^{t_{\ell-1},t_\ell}(\Lambda_0 L)\} - \frac{4\Delta t}{3} I_0^{t_{\ell-2},t_{\ell-1}}(\Lambda_0 N), \end{aligned} \tag{5.15}$$

$$\begin{aligned} S_{1,\ell} &= \sum_{r=1}^{m}\{I_{r0}^{t_{\ell-1},t_\ell}(\Lambda_r f) - \frac{2\Delta t}{3} I_r^{t_{\ell-1},t_\ell}(\Lambda_r L)\} + \sum_{r=1}^{m} I_{0r}^{t_{\ell-1},t_\ell}(\Lambda_0 g_r) \\ &+ \sum_{r,q=1}^{m} I_{qr}^{t_{\ell-1},t_\ell}(\Lambda_q g_r), \end{aligned} \tag{5.16}$$

and

(5.17)

$$S_{2,\ell-1} = \Delta t \sum_{r=1}^{m} \{ I_r^{t_{\ell-2},t_{\ell-1}}(\Lambda_r f) - \frac{2}{3} I_r^{t_{\ell-2},t_{\ell-1}}(\Lambda_r L) - \frac{4}{3} I_r^{t_{\ell-2},t_{\ell-1}}(\Lambda_r N)\}$$
$$-\frac{1}{3}\sum_{r=1}^{m} I_{r0}^{t_{\ell-2},t_{\ell-1}}(\Lambda_r f) - \frac{1}{3}\sum_{r=1}^{m} I_{0r}^{t_{\ell-2},t_{\ell-1}}(\Lambda_0 g_r) - \frac{1}{3}\sum_{r,q=1}^{m} I_{qr}^{t_{\ell-2},t_{\ell-1}}(\Lambda_q g_r).$$

Using conditions $II$ and $III$, we have from Lemma 2.12 in [7] that

$$\|S_{1,\ell}\|_{L_2} = O(\epsilon^2 \Delta t + \epsilon \Delta t^{3/2}) \tag{5.18}$$

and

$$\|S_{2,\ell-1}\|_{L_2} = O(\epsilon^2 \Delta t + \epsilon \Delta t^{3/2}) \tag{5.19}$$

which are of the desired order.

Next we further expand $R_\ell$. We have that $\Lambda_0 = \Lambda_0^f + \epsilon^2 \hat{\Lambda}_0$ and $R_\ell = R_\ell^f + \epsilon^2 \hat{R}_\ell$ where

$$R_\ell^f = \Delta t I_0^{t_{\ell-2},t_{\ell-1}}(\Lambda_0^f f) + I_{00}^{t_{\ell-1},t_\ell}(\Lambda_0^f f) - \frac{1}{3} I_{00}^{t_{\ell-2},t_{\ell-1}}(\Lambda_0^f f)$$
$$-\frac{2\Delta t}{3}\{I_0^{t_{\ell-2},t_{\ell-1}}(\Lambda_0^f L) + I_0^{t_{\ell-1},t_\ell}(\Lambda_0^f L)\} - \frac{4\Delta t}{3} I_0^{t_{\ell-2},t_{\ell-1}}(\Lambda_0^f N) \tag{5.20}$$

and

$$\hat{R}_\ell = \Delta t I_0^{t_{\ell-2},t_{\ell-1}}(\hat{\Lambda}_0 f) + I_{00}^{t_{\ell-1},t_\ell}(\hat{\Lambda}_0 f) - \frac{1}{3} I_{00}^{t_{\ell-2},t_{\ell-1}}(\hat{\Lambda}_0 f)$$
$$-\frac{2\Delta t}{3}\{I_0^{t_{\ell-2},t_{\ell-1}}(\hat{\Lambda}_0 L) + I_0^{t_{\ell-1},t_\ell}(\hat{\Lambda}_0 L)\} - \frac{4\Delta t}{3} I_0^{t_{\ell-2},t_{\ell-1}}(\hat{\Lambda}_0 N). \tag{5.21}$$

The term $\epsilon^2 \hat{R}_\ell$ is order $O(\epsilon^2 h^2)$ by Lemma 2.12 in [7] and thus we only need to concentrate on $R_\ell^f$.

Applying Itô's lemma to each term in the expression for $R_\ell^f$ and simplifying yields

$$R_\ell^f = \tilde{R}_\ell + \tilde{S}_{1,\ell} + \tilde{S}_{2,\ell-1} \tag{5.22}$$

where

(5.23)

$$\tilde{R}_\ell = \frac{\Delta t^2}{2} I_0^{t_{\ell-2},t_{\ell-1}}(\Lambda_0 \Lambda_0^f f) - \frac{2\Delta t^2}{3} I_0^{t_{\ell-2},t_{\ell-1}}(\Lambda_0 \Lambda_0^f L)$$
$$+\Delta t I_{00}^{t_{\ell-2},t_{\ell-1}}(\Lambda_0 \Lambda_0^f f) - \frac{2\Delta t}{3} I_{00}^{t_{\ell-2},t_{\ell-1}}(\Lambda_0 \Lambda_0^f L) - \frac{4\Delta t}{3} I_{00}^{t_{\ell-2},t_{\ell-1}}(\Lambda_0 \Lambda_0^f N)$$
$$-\frac{2\Delta t}{3} I_{00}^{t_{\ell-1},t_\ell}(\Lambda_0 \Lambda_0^f L) - \frac{1}{3} I_{000}^{t_{\ell-2},t_{\ell-1}}(\Lambda_0 \Lambda_0^f f) + I_{000}^{t_{\ell-1},t_\ell}(\Lambda_0 \Lambda_0^f f),$$

(5.24)

$$\tilde{S}_{1,\ell} = \epsilon \sum_{r=1}^{m} I_{r00}^{t_{\ell-1},t_\ell}(\hat{\Lambda}_r \Lambda_0^f f) - \epsilon \frac{2\Delta t}{3} \sum_{r=1}^{m} I_{r0}^{t_{\ell-1},t_\ell}(\hat{\Lambda}_r \Lambda_0^f L),$$

and

$$\tilde{S}_{2,\ell-1} = \epsilon\frac{\Delta t^2}{2}\sum_{r=1}^{m} I_r^{t_{\ell-2},t_{\ell-1}}(\hat{\Lambda}_r\Lambda_0^f f) - \epsilon\frac{2\Delta t^2}{3}\sum_{r=1}^{m} I_r^{t_{\ell-2},t_{\ell-1}}(\hat{\Lambda}_r\Lambda_0^f L)$$
$$+\epsilon\Delta t\sum_{r=1}^{m} I_{r0}^{t_{\ell-2},t_{\ell-1}}(\hat{\Lambda}_r\Lambda_0^f f) - \epsilon\frac{2\Delta t^2}{3}\sum_{r=1}^{m} I_{r0}^{t_{\ell-2},t_{\ell-1}}(\hat{\Lambda}_r\Lambda_0^f L)$$
$$-\epsilon\frac{4\Delta t^2}{3}\sum_{r=1}^{m} I_{r0}^{t_{\ell-2},t_{\ell-1}}(\hat{\Lambda}_r\Lambda_0^f N) - \frac{\epsilon}{3}\sum_{r=1}^{m} I_{r00}^{t_{\ell-2},t_{\ell-1}}(\hat{\Lambda}_r\Lambda_0^f f). \tag{5.25}$$

We then have that $L_\ell = R_\ell^\circ + S_{1,\ell}^\circ + S_{2,\ell-1}^\circ$ where

$$R_\ell^\circ = \tilde{R}_\ell + \epsilon^2\hat{R}_\ell, \tag{5.26}$$

$$S_{1,\ell}^\circ = S_{1,\ell} + \tilde{S}_{1,\ell}, \tag{5.27}$$

and

$$S_{2,\ell-1}^\circ = S_{2,\ell-1} + \tilde{S}_{2,\ell-1}. \tag{5.28}$$

The terms $S_{1,\ell}$, $S_{2,\ell-1}$ dominate the terms $\tilde{S}_{1,\ell}$, $\tilde{S}_{2,\ell-1}$ and from Lemma 2.12 in [7] we get the order estimates

$$\|R_\ell^\circ\|_{L_2} = O(\Delta t^3 + \epsilon^2\Delta t^2), \tag{5.29}$$

$$\|S_{1,\ell}^\circ\|_{L_2} = O(\epsilon^2\Delta t + \epsilon\Delta t^{3/2}), \tag{5.30}$$

and

$$\|S_{2,\ell-1}^\circ\|_{L_2} = O(\epsilon^2\Delta t + \epsilon\Delta t^{3/2}), \tag{5.31}$$

and additionally from Lemma 2.12 in [7] we have that (5.27) and (5.28) have zero expectation.

**6. Concluding Remarks.** We presented a new method for the approximation of stochastic differential equations. The method is easy to implement, requiring no evaluation of the derivatives of the drift or diffusion coefficients, and exhibits second order accuracy when applied to SDEs with small noise. Furthermore, the method allows for the efficient simulation of systems of the form (1.4) by reducing the time step restrictions associated with explicit methods. Because only linear terms in (1.4) are treated implicitly, fast linear solvers can be used alongside the method to keep computational cost to a minimum.

Additionally, we were able to prove that the method converges strongly, and hence weakly, under some assumptions of smoothness and growth of the drift and diffusion coefficients.

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