Complex Variables II: Homework 2

Read Chapter 10 in Stewart and Tall.

I. Exercises 10: #13, #15, #16, #19, #20

II.:

1. Assume f and g are analytic on a domain D. Prove that if $f(z)g(z) = 0$ for all $z \in D$, then either $f \equiv 0$ or $g \equiv 0$.

2. Assume f and g are entire and g never vanishes. If $|f(z)| \leq |g(z)|$ for all z, prove there exists a constant c such that $f(z) = cg(z)$. What happens if g has zeros?

3. Assume f is entire and $|f(z)| \to \infty$ as $z \to \infty$. Prove that f is a polynomial.

4. Let $\mathcal{F} = \{f : N_1(0) \to \mathbb{C} : f \text{ is analytic and } |f(z)| \leq 1 \text{ for all } z \in N_1(0)\}$. Let $\alpha \in \mathbb{C}$ be such that $|\alpha| < 1$. Prove that $\max_{f \in \mathcal{F}} |f'(\alpha)|$ is attained, and find its value.

5. Let $\{p_n(z)\}$ be a sequence of polynomials. Assume there exists $N \in \mathbb{N}$ such that degree$(p_n) \leq N$ for every n. If the sequence $\{p_n(z)\}$ converges uniformly on compact sets, prove that the limit is a polynomial $p(z)$ with degree$(p) \leq N$.

6. Assume f is analytic on $N_1(0)$ with $f(0) = 0$ and $f'(0) = 1$. Can you find an analytic function g (on a neighborhood of 0) such that $g(f(z)) = z$? (Hint: Find a Taylor series for g.)