Complex Variables II: Homework 3

Read Chapter 11 in Stewart and Tall.

I. Exercises 11: #1 (iii), (v), (vi); #2 (i, ii, iii) and (vi); #6, #14, #17, #22

II.:

1. Let f be analytic on the deleted neighborhood $D = \{z : 0 < |z - z_o| < R\}$ except at an infinite sequence of poles $\{a_n\}$ in D such that $a_n \to z_o$. Show that for any $w \in \mathbb{C}$ there is a sequence $\{z_n\}$ in D such that $z_n \to z_o$ and $f(z_n) \to w$.

2. Prove a stronger version of the Casorati-Weierstrass Theorem: Assume f has an essential singularity at z_o. Let $w \in \mathbb{C}$ and $\epsilon > 0$ be given. Then, for any $\delta > 0$, there exists $\alpha \in \mathbb{C}$ with $|w - \alpha| < \epsilon$, such that $f(z) = \alpha$ has infinitely many solutions in $N_{z_o}(\delta)$.

3. Show that an isolated singularity of $f(z)$ cannot be a pole of $\exp\{f(z)\}$.

4. The expression

$$\{f, z\} = \frac{f'''(z)}{f'(z)} - \frac{3}{2}\left(\frac{f''(z)}{f'(z)}\right)^2$$

is called the Schwarzian derivative of f. If f has an zero of order m, what is the smallest power of $z - z_o$ in the Laurent expansion of $\{f, z\}$? What if f has a pole of order m?

5. Assume f is analytic on $N_1(0)$ with $f(0) = 0$ and $f'(0) = 1$. Can you find an analytic function g (on a neighborhood of 0) such that $g(f(z)) = z$? (Hint: Find a Taylor series for g.)