Complex Variables II: Homework 4

Read Chapter 12 in Stewart and Tall.

I. Exercises 12: #2 (ii), (iii), (iv) and (v); #4; #6; #7, #14, #19; #26; #27

II.:
1. Let z_o be an isolated singularity of f. We saw in class that if $\lim_{z\to z_o}(z-z_o)f(z)=0$, then z_o is a removable singularity. For this problem, assume z_o is an isolated singularity of f, and let $D = N_R(z_o)\setminus\{z_o\}$ be a deleted neighborhood on which f is analytic. Prove that if

$$\iint_D |f(z)|^2 dx\, dy < \infty,$$

then z_o is a removable singularity. Hint: Using polar coordinates and the Cauchy integral formula, prove that for any analytic function h on a disc $N_\epsilon(w)$,

$$|h(w)|^2 \le \frac{1}{\pi\epsilon^2}\iint_{N_\epsilon(w)} |h(z)|^2\, dx\, dy.$$

Then, show that f cannot have an essential singularity or a pole.

2. Assume f and g are analytic on a disc $N_R(z_o)$. Assume the only zeros of g are at $z_1, z_2, ..., z_n$ and they are all of order 1. Compute

$$\frac{1}{2\pi i}\int_{C_R(z_o)} \frac{f(z)}{g(z)}\, dz$$

in terms of $f(z_1), ..., f(z_n)$ and $g'(z_1), ..., g'(z_n)$. What changes in your formula if the orders of the zeros can be greater than 1?

3. Assume f is analytic and bounded on the unit disc. Fix $|w|<1$. Show that

$$f(w) = \frac{1}{\pi}\iint_{N_1(0)} \frac{f(z)}{(1-\bar{z}w)^2}\, dx\, dy.$$

Hint: Start by rewriting the integral using polar coordinates.