Complex Variables II: Homework 5

Re-read Chapter 12 in Stewart and Tall. Turn in the following problems (the starred ones are optional.)

I. Exercises 12: #21, #22, #23, #24, #25*

II.:
1. Let f and g be analytic on $\overline{N_1(0)}$. Assume that the zeros of f are at $z_1, z_2, ..., z_k$ in $N_1(0)$ and that f has no zeros on the circle $C_1(0)$. Compute

$$\frac{1}{2\pi i}\int_{C_1}\frac{f'(z)}{f(z)}g(z)\,dz.$$

2. For each fixed integer n and each $0 \leq k \leq n$, give an example of a sequence of analytic functions $f_j : N_1(0) \to \mathbb{C}$ that converge uniformly on compact sets in the unit disc to a limit f_o with the following properties: Each f_j has at least n roots in the unit disc (counting multiplicities); but f_o has exactly k roots (counting multiplicities). What assumption can you add to guarantee that the limit f_o does have n roots?

3. (a) Prove that if $p(z)$ is a polynomial and all the zeros of $p(z)$ are contained in a disc $N_r(0)$, then all of the zeros of $p'(z)$ are also contained in $N_r(0)$. (Hint: Compute $\frac{p'(z)}{p(z)}$.)

(b) If $f(z)$ is an analytic function that has n zeros in a disc, can you say anything about the zeros of $f'(z)$?

4*. Prove that if f is analytic on $\overline{N_1(0)}$ and f is one-to-one on the boundary $C_1(0)$, then f is one-to-one on $\overline{N_1(0)}$.

5*. Assume that if f is analytic on a domain D and f is invertible on D. Fix $z_o \in \mathbb{D}$ and let $w_o = f(z_o)$. Prove that if $\delta > 0$ is sufficiently small, then

$$f^{-1}(w) = \frac{1}{2\pi i}\int_{C_\delta(z_o)}\frac{zf'(z)}{f(z)-w}\,dw$$

for all $w \in N_\epsilon(w_o)$ for some ϵ sufficiently small.
We can use this to find the same Taylor series expansion for $f^{-1}(w)$ that we found on a previous homework: Differentiate the above formula and show that

$$(f^{-1})'(w) = \frac{1}{2\pi i}\int_{C_\delta(z_o)}\left(\frac{1}{f(z)-w_o}\right)\left(1-\frac{w-w_o}{f(z)-w_o}\right)^{-1}dz.$$

Use a geometric series to expand the integrand, and integrate term-by term to obtain a sum for $(f^{-1})'(w)$. Let $f^{-1}(w) = \sum_{n=0}^{\infty} a_n(w-w_o)^n$, and find a formula for the coefficients a_n.