# Complex Variables II: Homework 6

Read Chapter 13 in Stewart and Tall.

**I. Exercises 13**: #1; #12; #13(i), (iii), (vi); #14; #15

**II.**:
1. Find a conformal map $f$ that maps the domain between the circles $|z| = 1$ and $|z - \frac{1}{4}| = \frac{1}{4}$ onto an annulus $a < |z| < 1$.

2. Find a conformal map $f$ between the domain $D$ onto the domain $S$.

   (i) $D = \{z = x + iy : -2 < x < 1\}$; $S = N_1(0)$

   (ii) $D = S = \{z = x + iy : y > 0\}$; also choose $f$ s.t. $f(-2) = -1$, $f(0) = 0$, $f(2) = 2$.

   (iii) $D = \{z = re^{i\theta} : r > 0 \text{ and } 0 < \theta < \frac{\pi}{4}\}$; $T = \{x + iy : 0 < y < 1\}$

   (iv) $D = N_1(0) \setminus \{z = x + iy : y = 0 \text{ and } x > 0\}$; $S = N_1(0)$

3. Suppose $f_1$ and $f_2$ are both conformal maps from a domain $D$ onto the unit disc. Also assume that for some $z_o \in D$, we know that $f_1(z_o) = f_2(z_o) = 0$ and $f_1'(z_o)$, $f_2'(z_o)$ are both real and strictly positive. Prove that $f_1 \equiv f_2$.

4. Fix any four distinct points $z_1, z_2, z_3, z_4$. Show you can find a value $k$ and a linear fractional transformation such that $z_1, z_2, z_3, z_4$ are mapped into $1, -1, k, -k$. How many solutions are there?

5. Let $z_1, z_2, z_3, z_4$ be four distinct points. Prove that the cross ratio $(z_1, z_2, z_3, z_4)$ is real if and only if the four points lie on a circle or on a straight line.

6. Let $a$ be a complex number with $|a| < 1$. Define

$$L(z) = \phi_a(z) = \frac{z - a}{1 - \bar{a}z}.$$

Consider the sequence $L_1 = L$; $L_{n+1} = L \circ L_n$ for $n \geq 1$. Assuming $L_\infty = \lim_{n\to\infty} L_n$ exists, find the function $L_\infty$.