Complex Variables II: Optional Final Homework

1. (a) Classify the types of isolated singularities of a function in terms of the limiting behavior of the function near the singularity.

(b) Let $R(z)$ be a rational function. Assume f is a meromorphic function, with poles at $a_1, a_2, ..., a_n$. Prove that if $|f(z)| \leq |R(z)|$ for all z at which both functions are defined, then $f \equiv cR$ for some constant $c \in \mathbb{C}$. (State the names of any theorems you use to prove this!)

2. Find the Taylor series of $\dfrac{z-1}{z^2}$ around $z_o = 1$. What is the radius of convergence? Outside the circle of convergence, find the Laurent series for this function.

3. Suppose f is analytic on some deleted neighborhood $N_r(0) \setminus \{0\}$. Let $g_k(z) = z^{-k} f(z)$. Prove that if $\text{Res}(g_k, 0) = 0$ for every $k \in \mathbb{Z}$, then $f \equiv 0$. Give an example of $f \not\equiv 0$ such that $\text{Res}(g_k, 0) = 0$ for every $k = 0, 1, 2, 3, ...$.

4. (a) Find the residue of $\dfrac{e^z}{(z-5)^4}$ at $z = 5$.

(b) Find the value of the integral $\displaystyle\int_C \frac{2z^3 - 3}{(z-3)(z^2+9)}\, dz$ around each of the following circles: (i) $|z - 2| = 2$, (ii) $|z + 2| = 2$, (iii) $|z| = 4$.

5. Calculate the following integral using residues (justify all your steps):

$$\int_0^\infty \frac{\cos(x) - 1}{x^2}\, dx$$

6. (a) Find a conformal map from the open unit square $(0,1) \times (0,1)$ to the region in the upper half plane intersected with some annulus $0 < r < |z| < R$. (You can choose the values of r and R.)

(b) Find a conformal map from the following infinite strip with a cut to the upper half plane: $S = \{z = x + iy \mid -1 < y < 1\} \setminus \{z = x + iy : y = 0 \text{ and } x \geq 0\}$.

7. State both Picard's little theorem and Picard's big theorem. Assume the big theorem and use it to prove the little theorem.