Math CS-120: Homework 5

Read Chapter 6 in Stewart and Tall.

I. Exercises 6: #2, #4, #5, #9, #10, #12, #15

II.:

1. Let $f(z) = |z|^2$. Use the Cauchy-Riemann equations to prove that f does not have an antiderivative. Compute the integrals $\int_\gamma f$ and $\int_\alpha f$ where γ is the path $[1, i]$ and α is the contour $[1, i+1] + [i+1, i]$.

2. Let $D \subseteq \mathbb{C}$ be a domain and let γ be a contour in D. If $f_n : D \to \mathbb{C}$ is a sequence of continuous functions such that $f_n \to f$ converges uniformly on D, show that

$$\int_\gamma f(z)\, dz = \lim_{n\to\infty} \int_\gamma f_n(z)\, dz.$$

3. Let $\gamma_r(t) = re^{it}$ for $t \in [0, \pi]$. Prove that

$$\lim_{r\to\infty} \int_{\gamma_r} \frac{e^{iz}}{z}\, dz = 0.$$

4. If f is a continuous, *real-valued* function on a domain containing the circle $|z| = 1$, and if $|f(z)| \le 1$, prove that

$$\left| \int_\gamma f \right| \le 4.$$

(where γ is a simple, closed path whose image is the unit circle.)

5. Assume $f(z)$ is analytic (and that $f'(z)$ is continuous) in a domain D. Let γ be a *closed* path in D.

   (a) Show that $\int_\gamma \overline{f(z)} f'(z)\, dz$ is imaginary.

   (b) Show that, if $|f(z) - 1| < 1$ for all $z \in D$, then $\int_\gamma \frac{f'(z)}{f(z)}\, dz = 0$.

6. We can define integration with respect to $d\bar{z}$ by $\int_\gamma f(z)\, d\bar{z} = \overline{\int_\gamma \overline{f(z)}\, dz}$
This allows us to define regular path integrals in terms of complex integrals by the formulas

$$\int_\gamma f\, dx = \frac{1}{2}\left(\int_\gamma f\, dz + \int_\gamma f\, d\bar{z}\right); \quad \int_\gamma f\, dy = \frac{1}{2i}\left(\int_\gamma f\, dz - \int_\gamma f\, d\bar{z}\right).$$

Let $p(z)$ be a polynomial, and γ be a simple, closed path whose image is the (counter-clockwise oriented) circle $|z - z_o| = r$. What is the value of $\int_\gamma p(z)\, d\bar{z}$?