Math CS-120: Homework 6

Read Chapter 7 in Stewart and Tall.

I. Exercises 7: #5, #6, #9, #13, #15,
#17 (show the computation for $\cosh^{-1}$ only; you may want to review the computation we did in class for $\cos^{-1}$ first)
#18, #20, #22

II.:
1. Give an example of a closed contour γ such that, for every integer k, there exists a point $a \notin Im(\gamma)$ such that $\omega(\gamma, a) = k$.

2. Prove Cauchy's theorem on a rectangle under the slightly weaker hypothesis:
Let f be analytic on a rectangle R, except at finitely many points $\xi_1, \xi_2, ..., \xi_n \in R^o$ (the interior of R). If we know that

$$\lim_{z \to \xi_j} (z - \xi_j) f(z) = 0 \quad \text{for each } 1 \leq j \leq n,$$

then $\displaystyle\int_{\partial R} f = 0$.

3. Let $p(z)$ be a polynomial and γ be a closed path (along which $p(z) \neq 0$). Prove that

$$\int_\gamma \frac{p'(z)}{p(z)}\, dz = 2\pi i n.$$

for some integer n. [*Hint: For a simple way to do this, look at the first proof we did in class to show that the integral definition of the winding number always returns an integer!*]

4. Let γ be a closed contour in $\mathbb{C} \setminus \{0\}$. Consider points z_1 and z_2 on γ. Denote the contour that traces out γ starting at z_1 and ending at z_2 by γ_1; denote the contour along γ from z_2 to z_1 by γ_2. (So, we have that $\gamma = \gamma_1 + \gamma_2$.) Assume that $\text{Im } z_1 < 0$ and $\text{Im } z_2 > 0$. If γ_1 does not intersect the negative real axis and γ_2 does not intersect the positive real axis, prove that $\omega(\gamma, 0) = 1$.

5. The Jordan curve theorem states that a simple closed curve defines exactly two connect components. We have already seen that a closed curve always defines exactly one unbounded component, and that $\omega(\gamma, z) = 0$ for every z in this domain. Let γ be a simple closed curve; we can use the winding number to show that there exist a second connected component by proving there exists a point a such that $\omega(\gamma, a) \neq 0$. (Unfortunately, however, this doesn't show there are *exactly* two connected components.) [*Hint: By translation, you may assume that the curve lies entirely in the right half-plane and that it intersects the real axis.*]