Math CS-120: Homework 7

Read Chapters 8 and 9 in Stewart and Tall.

I. Exercises 8: #5, #7

II. Exercises 9: #2, #4, #5, #7, #8, #10, #11

III.:
1. Let γ be a closed contour. Suppose ϕ is a continuous function on γ. Prove that, for all $n \geq 1$,

$$F_n(z) = \int_\gamma \frac{\phi(w)}{(w-z)^n}\,dw$$

is analytic for all z. (Hint: Show that $F_n'(z) = nF_{n+1}(z)$.)
(*Note that this rigorously proves that, for any function f analytic on a general domain, f is infinitely differentiable on the domain! This is because we have already proven Cauchy's integral formula, which implies the representation formula $f(z) = \frac{1}{2\pi i}\int_C \frac{f(w)}{(w-z)}\,dw$ (for z inside C, where C is a circle inside a disc on which f is analytic.*)

2. More generally than #1: Let D be a domain and $\phi(z,t)$ be a continuous function on $D \times [a,b]$. Assume that $\phi_t(z) = \phi(z,t)$ is analytic on D for each fixed t. Prove that

$$F(z) = \int_a^b \phi(z,t)\,dt$$

is analytic on D, and that

$$F'(z) = \int_a^b \phi_t'(z)\,dt.$$

(*Hint: Use the representation formula to write ϕ_t as an integral.*)

3. Define

$$f(z) = \frac{1}{\pi}\int_0^1 r\left(\int_{-\pi}^{\pi} \frac{1}{re^{i\theta}+z}\,d\theta\right)dr$$

Show that for $|z| < 1$, $f(z) = \bar{z}$ and that for $|z| \geq 1$, $f(z) = \frac{1}{z}$.
Notice that f is continuous, but not analytic! Why doesn't this contradict #2? (*Hint: What's $\phi(z,r)$ in this integral?*)