Math 108b: Notes 1/14/11

DEFINITIONS:

The $\tilde{A}_{ij}$ **minor** of an $n \times n$ matrix A is defined to be the $(n-1) \times (n-1)$ matrix defined by removing the i^{th} row and the j^{th} column of A.

The **determinant** of a matrix is a recursively defined function that takes $n \times n$ matrices to scalars:

(i) For $n = 1$, define, for any matrix $A = (a_{11})$, $\det A = a_{11}$.

(ii) Once the determinant is defined for all $(n-1) \times (n-1)$ matrices, define it for all $n \times n$ matrices A by

$$\det A = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det \tilde{A}_{1j}$$

Notice this gives the right definition for 2×2 matrices! As you know, you can think of this definition as an expansion along the first row of A; we can eventually prove that expanding along any *row or column gives the same result.*

THEOREM 4.3 The determinant is an $n-$**linear** function.

This means that for every $n \times n$ matrix A over the field F, which we can write as

$$A = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix},$$

where the a_i are (row) vectors in F^n, the determinant is linear in each row. In other words, if $a_r = u + kv$ for some $1 \leq r \leq n$ and some $u, v \in F^n$ and some scalar $k \in F$, then

$$(\star) \qquad \det A = \det \begin{pmatrix} a_1 \\ \vdots \\ a_{r-1} \\ u + kv \\ a_{r+1} \\ \vdots \\ a_n \end{pmatrix} = \det \begin{pmatrix} a_1 \\ \vdots \\ a_{r-1} \\ u \\ a_{r+1} \\ \vdots \\ a_n \end{pmatrix} + k \det \begin{pmatrix} a_1 \\ \vdots \\ a_{r-1} \\ v \\ a_{r+1} \\ \vdots \\ a_n \end{pmatrix}.$$

Proof by induction:

<u>Base case</u>: $n = 1$. Let $A \in M_{1\times 1}(F)$ be given. In this case, $A = (a_1)$, where $a_1 \in F^1$. Suppose that $a_1 = u + kv$ for some $u, v \in F^1$ and some scalar $k \in F$. Then,

$$\det A = a_1 = u + kv = \det(u) + k \det(v).$$

Therefore the result $(\star)$ holds for all 1×1 matrices.

Inductive step: Assume (IH): for every $A \in M_{(n-1)\times(n-1)}(F)$, with rows $a_1, a_2, ..., a_{n-1}$, if $1 \le r \le n-1$ and $a_r = u + kv$ for some $u, v \in F^{n-1}$ and some $k \in F$, then the result $(\star)$ holds.

Now, we shall prove the result for all $n \times n$ matrices. Let $A \in M_{n\times n}(F)$. Write the rows of A as $a_1, a_2, ..., a_n \in F^n$. Suppose that for some r, $1 \le r \le n$, $a_r = u + kv$ where $u, v \in F^n$ are (row) vectors and $k \in F$. (Notice this means we can write $u = (u_1, u_2, ..., u_n)$ and $v = (v_1, v_2, ..., v_n)$ for $u_i, v_i \in F$.) We want to show $(\star)$; defining the matrices

$$B = \begin{pmatrix} a_1 \\ \vdots \\ a_{r-1} \\ u \\ a_{r+1} \\ \vdots \\ a_n \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} a_1 \\ \vdots \\ a_{r-1} \\ v \\ a_{r+1} \\ \vdots \\ a_n \end{pmatrix},$$

$(\star)$ becomes $\det A = \det B + k \det C$. We prove this for the two cases $r = 1$ and $r \ne 1$:

Case 1. $r = 1$.
In this case, the matrices A, B, and C only differ in the first row. Therefore, the minor defined by removing the first row are all the same: That is, $\tilde{A}_{1j} = \tilde{B}_{1j} = \tilde{C}_{1j}$ for each $1 \le j \le n$. Note that the first row $a_1 = u + kv = (u_1 + kv_1, u_2 + kv_2, ..., u_n + kv_n)$, so $a_{1j} = u_j + kv_j$. Then, computing from the definition of determinant,

$$\begin{aligned} \det A &= \sum_{j=1}^{n} (-1)^{1+j}(u_j + kv_j) \det \tilde{A}_{1j} = \sum_{j=1}^{n} (-1)^{1+j} u_j \det \tilde{A}_{1j} + \sum_{j=1}^{n} (-1)^{1+j} kv_j \det \tilde{A}_{1j} \\ &= \sum_{j=1}^{n} (-1)^{1+j} u_j \det \tilde{B}_{1j} + k \sum_{j=1}^{n} (-1)^{1+j} v_j \det \tilde{C}_{1j} = \det B + k \det C, \end{aligned}$$

since the two sums above represent exactly the definitions of $\det B$ and $\det C$, expanding along the first rows (note the first row of B is the vector u, and the first row of C is the vector v).

Case 2. $1 < r \le n$.
In this case, note that for each fixed $1 \le j \le n$, we can write the minor $\tilde{A}_{1j}$ as follows:

$$\tilde{A}_{1j} = \begin{pmatrix} \tilde{a}_2 \\ \vdots \\ \tilde{a}_{r-1} \\ \tilde{u} + k\tilde{v} \\ \tilde{a}_{r+1} \\ \vdots \\ \tilde{a}_n \end{pmatrix}$$

where $\tilde{a}_s = (a_{s1}, ..., a_{s(j-1)}, a_{s(j+1)}, ..., a_n)$ – that is, remove the j^{th} entry in the column a_s. (Similarly, define $\tilde{u} = (u_1, ..., u_{j-1}, u_{j+1}, ..., u_n)$ and $\tilde{v} = (v_1, ..., v_{j-1}, v_{j+1}, ..., v_n)$). If B is the matrix corresponding to replacing the r^{th} row of A by u and C is the matrix corresponding to replacing the r^{th} row of A by v (as above), then clearly removing the 1^{st} row and j^{th} column yields the minors

$$\tilde{B}_{1j} = \begin{pmatrix} \tilde{a}_2 \\ \vdots \\ \tilde{a}_{r-1} \\ \tilde{u} \\ \tilde{a}_{r+1} \\ \vdots \\ \tilde{a}_n \end{pmatrix} \quad \text{and} \quad \tilde{C}_{1j} = \begin{pmatrix} \tilde{a}_2 \\ \vdots \\ \tilde{a}_{r-1} \\ \tilde{v} \\ \tilde{a}_{r+1} \\ \vdots \\ \tilde{a}_n \end{pmatrix}$$

Since $\tilde{A}_{1j}$ is an $(n-1)\times(n-1)$ matrix with r^{th} row $\tilde{a}_r = \tilde{u} + k\tilde{v}$, (IH) implies that

$$\det \tilde{A}_{1j} = \det \tilde{B}_{1j} + k \det \tilde{C}_{1j}.$$

Therefore, computing,

$$\begin{aligned} \det A &= \sum_{j=1}^{n} (-1)^{1+j}(a_{1j}) \det \tilde{A}_{1j} = \sum_{j=1}^{n} (-1)^{1+j} a_{1j}(\det \tilde{B_{1j}} + k \det \tilde{C_{1j}}) \\ &= \sum_{j=1}^{n} (-1)^{1+j} u_j \det \tilde{B}_{1j} + k \sum_{j=1}^{n} (-1)^{1+j} v_j \det \tilde{C}_{1j} = \det B + k \det C. \quad \square \end{aligned}$$

THEOREM 4.5 The determinant is an **alternating** function.

This means that interchanging any two rows of a matrix changes its determinant by a minus sign. In other words, given any $A \in M_{n\times n}(F)$, fix $1 \le r < s \le n$. Writing the rows of A as $a_1, ..., a_r, ..., a_s, ..., a_n$,

$$\det A = \det \begin{pmatrix} a_1 \\ \vdots \\ a_r \\ \vdots \\ a_s \\ \vdots \\ a_n \end{pmatrix} = -\det \begin{pmatrix} a_1 \\ \vdots \\ a_s \\ \vdots \\ a_r \\ \vdots \\ a_n \end{pmatrix}$$

Proof: A full proof is given in the book (using the results of the theorems leading up to Theorem 4.5). As an exercise, you should be able to write down a direct inductive proof of this result, at least for the case $r \neq 1$!

Notice that the above theorem directly implies that, if two rows of A are identical, then $\det A = 0$. We can extend this to say that if B is the matrix obtained adding a multiple of one row of A to another row of A (in other words, $B = EA$, for E an elementary matrix of type 3!), then $\det B = \det A$:

Proof: Given $A \in M_{n\times n}(F)$, write the rows of A as $a_1, a_2, ..., a_n$. Now, suppose B has rows $a_1, ..., a_{s-1}, a_s + ka_r, a_{s+1}, ..., a_n$, for some $1 \le r, s \le n$ $(r \neq s)$ and some $k \in F$.

Then, by the n-linearity of the determinant:

$$\det B = \det \begin{pmatrix} a_1 \\ \vdots \\ a_{s-1} \\ a_s + k a_r \\ a_{s+1} \\ \vdots \\ a_n \end{pmatrix} = \det \begin{pmatrix} a_1 \\ \vdots \\ a_{s-1} \\ a_s \\ a_{s+1} \\ \vdots \\ a_n \end{pmatrix} + k \det \begin{pmatrix} a_1 \\ \vdots \\ a_{s-1} \\ a_r \\ a_{s+1} \\ \vdots \\ a_n \end{pmatrix} = \det A + 0$$

since the second matrix has two identical rows (namely, a_r must appear twice!) □

Therefore, since the determinant is alternating and n-linear, we have that:

- for elementary matrices of type 1, $\det E = -\det I = -1$
  (recall, for type 1, E is the identity matrix with two rows exchanged)
- for elementary matrices of type 2, $\det E = k \det I = k$
  (*where $k \neq 0$ is the constant multiplying one of the rows of the identity matrix*)
- for elementary matrices of type 3, $\det E = \det I = 1$.