From section 6.3, we know that every linear operator $T$ on a finite dimensional inner product space $V$ has an adjoint. ($T^*$ is defined as the unique linear operator on $V$ such that $\langle T(x), y \rangle = \langle x, T^*(y) \rangle$ for every $x, y \in V$ – see Theorems 6.8 and 6.9.) When $V$ is infinite dimensional, the adjoint $T^*$ may or may not exist.

One useful fact (Theorem 6.10) is that if $\beta$ is an orthonormal basis for a finite dimensional inner product space $V$, then $[T^*]_\beta = [T]_\beta^\dagger$. That is, the matrix representation of the operator $T^*$ is equal to the complex conjugate of the matrix representation for $T$.

For a general vector space $V$, and a linear operator $T$, we have already asked the question “when is there a basis of $V$ consisting only of eigenvectors of $T$?” – this is exactly when $T$ is diagonalizable. Now, for an inner product space $V$, we know how to check whether vectors are orthogonal, and we know how to define the norms of vectors, so we can ask “when is there an orthonormal basis of $V$ consisting only of eigenvectors of $T$?” Clearly, if there is such a basis, $T$ is diagonalizable – and moreover, eigenvectors with distinct eigenvalues must be orthogonal.

**Definitions** Let $V$ be an inner product space. Let $T \in \mathcal{L}(V)$.

(a) $T$ is **normal** if $T^*T = TT^*$

(b) $T$ is **self-adjoint** if $T^* = T$

For the next two definitions, assume $V$ is finite-dimensional: Then,

(c) $T$ is **unitary** if $F = \mathbb{C}$ and $\|T(x)\| = \|x\|$ for every $x \in V$

(d) $T$ is **orthogonal** if $F = \mathbb{R}$ and $\|T(x)\| = \|x\|$ for every $x \in V$

**Notes**

1. Self-adjoint $\Rightarrow$ Normal (Clearly, if $T^* = T$, then $T^*$ and $T$ commute!)

2. Unitary or Orthogonal $\Rightarrow$ Invertible (Show that $N(T) = \{0\}$.)

3. Unitary or Orthogonal $\Rightarrow$ Normal (This is because, when $T$ is unitary (or orthogonal), we must have $T^*T = TT^* = I$, the identity operator on $V$ – see Theorem 6.18.)
A simple example of an orthogonal operator is the rotation operator $T_\theta$ on $\mathbb{R}^2$. In the standard basis, 

$$A_\theta = [T_\theta]_\beta = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}.$$ 

Notice that $A$ is orthogonal. Also, $A^* = A^T \neq A$, so $A$ is not self-adjoint. However, it is easy to check that $AA^* = A^*A = I$, so $A$ is normal. If the matrix $A$ were over the complex field, it would have an orthonormal basis of eigenvectors by the spectral theorem (see below)! However, over the reals, we see that the characteristic polynomial of $A$ does not split (unless $\sin \theta = 0$) – therefore, there are no real eigenvalues and therefore no eigenvectors at all for the operator $T_\theta$ on $\mathbb{R}^2$.

Spectral Theorem 1 Let $T$ be a linear operator on a finite dimensional complex inner product space $V$. Then, $T$ is normal if and only if there exists an orthonormal basis (for $V$) consisting of eigenvectors of $T$.

Spectral Theorem 2 Let $T$ be a linear operator on a finite dimensional real inner product space $V$. Then, $T$ is self-adjoint if and only if there exists an orthonormal basis (for $V$) consisting of eigenvectors of $T$.

We’ll prove the simple direction first: Assume that there is an orthonormal basis $\beta$ consisting of eigenvectors of $T$. Then, we know that $[T]_\beta$ is diagonal, and also that $[T^*]_\beta = [T]_\beta^*$ is diagonal. Since diagonal matrices commute, we have that 


Hence, $T^*T = TT^*$, so $T$ is normal. This holds whether $F = \mathbb{C}$ or $F = \mathbb{R}$.

We’ll see in the lemmas below that $T$ normal implies that every eigenvector of $T$ with eigenvalue $\lambda$ is also an eigenvector of $T$ with eigenvalue $\overline{\lambda}$. Since $\beta$ is a basis of eigenvectors of either $T$ or $T^*$, we have that both $[T]_\beta$ and $[T^*]_\beta$ are diagonal matrices with their eigenvalues on the diagonal. In the case $F = \mathbb{R}$, every eigenvalue of $T$ is real, so $\lambda = \overline{\lambda}$, and we must have that $[T^*]_\beta = [T]_\beta$. Therefore, in the case of a real inner product space, we know that $T$ is not only normal, but also self-adjoint!

Lemma (pg. 269) Let $V$ be a finite-dimensional inner product space and $T \in \mathcal{L}(V)$. If $T$ has an eigenvector with eigenvalue $\lambda$, then $T^*$ has an eigenvector with eigenvalue $\overline{\lambda}$.

See the book for the proof. Note that the above lemma is true for any linear operator $T$ – but we do not know that the eigenvectors for $T$ and $T^*$ are related! The next theorem states that when $T$ is normal, we know the eigenvectors for $T$ and $T^*$ are the same – this is the fact we used in the proof of the spectral theorem above.
Theorem 6.15 (pg. 371) Let $V$ be an inner product space and $T \in \mathcal{L}(V)$ be a normal operator. Then,

(c) If $T(x) = \lambda x$ (for some $x \in V$ and some $\lambda \in F$), then $T^*(x) = \bar{\lambda}x$.

(d) If $\lambda_1$ and $\lambda_2$ are distinct eigenvalues of $T$ with eigenvectors $x_1$ and $x_2$, then $\langle x_1, x_2 \rangle = 0$.

Sketch of proof: First prove that, since $T$ is normal, $\|T(x)\| = \|T^*(x)\|$ for every $x \in V$. Then, since $U = T - \lambda I$ is normal, and $U^* = T^* - \bar{\lambda}I$, we know that $U(x) = 0$ if and only if $U^*(x) = 0$. In other words, $T(x) = \lambda x$ if and only if $T^*(x) = \bar{\lambda}I$.

For (d), Simply compute $\lambda_1 \langle x_1, x_2 \rangle = \lambda_2 \langle x_1, x_2 \rangle$. Since $\lambda_1 \neq \lambda_2$, we must have $\langle x_1, x_2 \rangle = 0$.

We’re almost ready to finish the proof of the first spectral theorem, but first, we’ll need the following theorem:

Schur’s Theorem Let $T$ be a linear operator on a finite-dimensional inner product space $V$. Suppose the characteristic polynomial of $T$ splits. Then, there exists an orthonormal basis $\beta$ for $V$ such that the matrix $[T]_{\beta}$ is upper triangular.

Proof: Induct on $n = \dim V$. Of course, if $n = 1$, take any basis $\beta = \{x\}$ with $\|x\| = 1$. Then, $[T]_{\beta} = (a)$ (where $a = T(x)$) is upper-triangular. Now, assume the result for $n - 1$: that is, suppose that if $V$ is any finite-dimensional inner product space with dimension $n - 1$ and if $T$ is any linear operator such that the characteristic polynomial splits, then there exists an orthonormal basis for $V$ such that $[T]_{\beta}$ is upper triangular. We wish to show this result for $n$.

Fix an $n-$dimensional inner product space $V$ and a $T \in \mathcal{L}(V)$ such that the characteristic polynomial of $T$ splits. This implies that $T$ has an eigenvalue $\lambda$. Therefore, $T^*$ has an eigenvalue $\bar{\lambda}$ (by Lemma, pg. 369). Let $z$ be a unit eigenvector of $T^*$, so $T^*z = \bar{\lambda}z$. Consider the space $W = \text{span}\{z\}$. Clearly, $W^\perp$ is $n - 1$ dimensional, and $T_{W^\perp}$ is a linear operator on $W^\perp$, so by the induction hypothesis, there exists a basis $\gamma$ such that $[T_{W^\perp}]_{\gamma}$ is upper-triangular. (Notice that we know that $T_{W^\perp}$ splits since $W^\perp$ is $T$–invariant! Check this fact below.)

Let $y \in W^\perp$. We want to show that $T(y)$ is in $W^\perp$. For every ______, compute

Finally $\beta = \gamma \cup \{z\}$ is orthonormal, and $[T]_{\beta}$ is upper-triangular: Letting $[Tz]_{\beta} = a \in F^n$,

$$[T]_{\beta} = \begin{pmatrix} [T_{W^\perp}]_{\gamma} & | & a \\ 0 & | & 0 \end{pmatrix}. \quad \square$$


Proof of Spectral Theorem 1 Assume $T$ is normal. Since $F = \mathbb{C}$, we know (from the fundamental theorem of algebra) that the characteristic polynomial of $T$ splits. By Schur’s theorem, we can find an orthonormal basis $\beta = \{v_1, v_2, \ldots, v_n\}$ such that $A = [T]_\beta$ is upper-triangular. Clearly $v_1$ is an eigenvector of $T$ (since $T(v_1) = A_{11}v_1$.) The fact that $T$ is normal will imply that $v_2, \ldots, v_n$ are also eigenvectors!

Assume $v_1, v_2, \ldots, v_{k-1}$ are all eigenvectors Let $\lambda_1, \lambda_2, \ldots, \lambda_{k-1}$ be the corresponding eigenvalues. Then, since $A$ is upper-triangular, look at the $k^{th}$ column of $A$ and note $A_{mk} = 0$ if $m > k$: This lets us compute

$$T(v_k) = A_{1k}v_1 + A_{2k}v_2 + \ldots A_{kk}v_k$$

From Theorem 6.15, the fact that $T$ is normal implies that $T^*(v_j) = \bar{\lambda}_j v_j$. Since the basis is orthonormal, we have the formula $A_{jk} = \langle T(v_k), v_j \rangle$ for the coefficients in the equation above. Computing,

$$A_{jk} = \langle T(v_k), v_j \rangle = \langle v_k, T^*(v_j) \rangle = \langle v_k, \bar{\lambda}_j v_j \rangle = \lambda_j \langle v_k, v_j \rangle = 0.$$ 

Therefore, we have simply that $T(v_k) = A_{kk}v_k$, so $v_k$ is an eigenvector for $T$.

By induction, we have shown that $v_1, v_2, \ldots, v_n$ are all eigenvectors of $T$, so $\beta$ is an orthonormal basis consisting of only eigenvectors of $T$, and the spectral theorem is proven. □

Before we can prove the second version of the spectral theorem, for $F = \mathbb{R}$, we need the following lemma:

LEMMa (pg. 373) Let $T$ be a self-adjoint operator on a finite-dimensional inner product space $V$. Then the following two facts hold (whether we have $F = \mathbb{R}$ or $F = \mathbb{C}$)

(a) Every eigenvalue of $T$ is real.

(b) The characteristic polynomial of $T$ splits.

Proof of (a): From Theorem 6.15, if $x$ is an eigenvalue of $T$, we have both $T(x) = \lambda x$ for some $\lambda \in F$ and $T^*(x) = \bar{\lambda} x$. However, since $T$ is self-adjoint, $T(x) = T^*(x)$. Therefore, $\lambda x = \bar{\lambda} x$. Since $x \neq 0$, we have that $\lambda = \bar{\lambda}$; i.e., $\lambda$ is real.

Proof of (b): When $F = \mathbb{C}$, we already know the characteristic polynomial splits. Let $\beta$ be any orthonormal basis for $V$; we know that $A = [T]_\beta$ satisfies $A^* = A$. Define, for every $x \in \mathbb{C}^n$,

$$S(x) = Ax.$$ 

$S$ has the same characteristic polynomial as $T$, but since $S$ is a linear operator on a complex inner product space, we know the characteristic polynomial of $S$ splits. Moreover, since $S$ is self-adjoint, each eigenvalue $\lambda_1, \ldots, \lambda_n$ is real, and the polynomial $p_T(t) = p_S(t) = (\lambda_1 - t)(\lambda_2 - t)\ldots(\lambda_n - t)$ splits over the field $F = \mathbb{R}$. □
Proof of Spectral Theorem 2: Assume $F = \mathbb{R}$ and $T$ is a self-adjoint linear operator on $V$. Then, the characteristic polynomial of $T$ splits, and Schur’s theorem implies that there exists an orthonormal basis $\beta$ for $V$ such that $A = [T]_\beta$ is upper-triangular. The same proof as for the first spectral theorem now works since $T^* = T$, we know that both $A$ and $A^T = A^* = A$ are upper triangular. Therefore, $A$ is diagonal.

Finally, we will show the next theorem, which includes the fact that unitary (and orthogonal) operators must be normal!

**Theorem 6.18** Let $V$ be a finite dimensional inner product space and $T \in \mathcal{L}(V)$. The following are equivalent:

(a) $T^*T = TT^* = I$

(b) $\langle T(x), T(y) \rangle = \langle x, y \rangle$ for every $x, y \in V$

(c) $\|T(x)\| = \|x\|$ for every $x \in V$

**Proof:**

(a) $\Rightarrow$ (b) Assume (a) $T^*T = TT^* = I$.

Given $x, y \in V$, since $T^*T(x) = I(x) = x$, we have $\langle x, y \rangle = \langle T^*T(x), y \rangle = \langle T(x), T(y) \rangle$.

(b) $\Rightarrow$ (e) Assume (b) $\langle x, y \rangle = \langle T(x), T(y) \rangle$ for every $x, y \in V$

Let $\beta = \{v_1, v_2, ..., v_n\}$ be an orthonormal basis for $V$. Then $T(\beta) = \{T(v_1), T(v_2), ..., T(v_n)\}$ is an orthonormal basis for $V$ (since, from (b), $\langle T(v_i), T(v_j) \rangle = \langle v_i, v_j \rangle = \delta_{ij}$).

Given any $x \in V$, write $x = \sum_{i=1}^{n} a_i v_i$, we can easily compute $\|x\|^2 = \sum_{i=1}^{n} |a_i|^2$.

Since $T$ is linear, $T(x) = \sum_{i=1}^{n} a_i T(v_i)$; the same computation shows $\|T(x)\|^2 = \sum_{i=1}^{n} |a_i|^2$. 
(e) ⇒ (a) Assume (e) $\|T(x)\|^2 = \|x\|^2$ for every $x \in V$.

For all $x \in V$,

$$\langle x, x \rangle = \|x\|^2 = \|T(x)\|^2 = \langle T(x), T(x) \rangle = \langle x, T^*T(x) \rangle$$

This implies, for all $x \in V$, that

$$\langle x, (I - T^*T)(x) \rangle = 0.$$  \hspace{1cm} (\star)

Let $U = I - T^*T$. Check that $U$ is self-adjoint: $U^* = I^* - (T^*T)^* = I - T^*T$. Then, by the spectral theorem there exists an orthonormal basis $\beta$ consisting of eigenvectors of $U$.

For all $x \in \beta$, $U(x) = \lambda x$. By (\star), $0 = \langle x, U(x) \rangle = \bar{\lambda} \langle x, x \rangle$. Since $x \neq 0$, we must have $\lambda = 0$. This implies that $[U]_\beta$ is the zero matrix! Hence, $U = I - T^*T = T_o$ (the zero transformation on $V$), so $T^*T = I$. Since $V$ is finite dimensional, this proves that $T^*$ must be the inverse of $T$ and therefore $TT^* = I$. $\square$