Math 108b: Review of 108a

Vector Spaces, Basis and Dimension

What is a *vector space*? We can think of it as an abstraction of the set of vectors we are used to drawing in the plane $\mathbb{R}^2$. The important properties of vectors are that we can ____________ them together and we can ________________ a vector by any ______________________. It is the algebraic properties of these operations that we abstract: every vector space is required to satisfy them.

DEFINITION A **vector space** V over a *field* F (in examples, often $\mathbb{R}$ or $\mathbb{C}$) is defined by the set of vectors V and ____________________________________ that satisfy:

(1) Addition is commutative. ____

(2) Addition is associative. ____

(3) 0 exists. ____

(4) Additive inverses exist. ____

Multiplication by scalars:
- (5) For all $x \in V$, ______________________________
- (6) For all $x \in V$ and for all $a, b \in F$, ____________

Distributive laws:
- (7) ____
- (8) ____

The picture of vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$ is very useful to understand vector spaces and gain some geometric intuition for and understanding of theorems, but there are many other natural examples of vector spaces. For instance, the set of $n \times m$ matrices with real entries ($M_{n\times m}(\mathbb{R})$)or the set of polynomials with complex coefficients $\mathcal{P}(\mathbb{C})$.

DEFINITION Let V be a vector space over F and W be a subset of V. W is called a **subspace** of V if ____.

THEOREM Let V be a vector space and W be a subset of V. W is a subspace of V if and only if

(a) ____________

(b) ____

(I.e., _________________________ and ______________________________________.)

Examples: The subset of symmetric matrices is a subspace of $M_{n\times m}(F)$ and the subset of polynomials of degree less than or equal to n (denoted by $\mathcal{P}_n(F)$) is a subspace of $\mathcal{P}(F)$.

DEFINITION A subset S of a vector space V is **linearly dependent** if ____________ distinct vectors $u_1, u_2, ..., u_n \in S$ and scalars $a_1, a_2, ..., a_n$, at least one of which is ________, such that

____.

(*In other words, S is linearly dependent if there is a nontrivial* ____________________ *of* 0 *as a* ______________________________ *of vectors in the set S.*)

DEFINITION A subset S of a vector space V that is not linearly dependent is called **linearly independent**.

DEFINITION A **basis** β for a vector space V is a linear independent subset of V that *generates* V. (I.e., __________________.)

THEOREM A set $\beta = \{u_1, u_2, ..., u_n\}$ is a basis for V if and only if every $v \in V$ there is a unique list of scalars $a_1, a_2, ..., a_n \in F$ such that

____.

THEOREM If V has a finite basis, then every basis of V has the same number of elements. In this case, V is called **finite-dimensional** and the **dimension** of V is defined to be the cardinality of any basis of V.

Finite-dimensional vector spaces – by definition! – must have a basis. What about infinite dimensional ones? Using the axiom of choice, it can be shown that every vector space has a basis.

Linear Transformations

DEFINITION Let V and W be vector spaces over F. A function $T : V \to W$ is a **linear transformation** from V to W if, for all $x, y \in V$ and $c \in F$, both (i) and (ii) hold:

(i) ______________________________

(ii) ______________________________

Some examples of linear transformations: the identity map, the zero transformation, differentiation of polynomials (this is a map from $\mathcal{P}(F)$ to $\mathcal{P}(F)$).

DEFINITION Let $T : V \to W$ be a linear transformation from a vector space V to a vector space W. The **null space** (or **kernel**) of T, $N(T) =$ __________________________________.

We can prove that for every linear transformation $T : V \to W$, $N(T)$ is a ______________ of V. If $N(T)$ is finite-dimensional, we define the **nullity** of T to be the dimension of $N(T)$.

Definition Let $T : V \to W$ be a linear transformation from a vector space V to a vector space W. The **range** (or **image**) of T, $R(T) =$ ________________________________.

We can prove that for every linear transformation $T : V \to W$, $R(T)$ is a ____________ of W. If $R(T)$ is finite-dimensional, we define the **rank** of T to be the dimension of $R(T)$.

Theorem **Dimension Theorem**
Let V and W be vector spaces and $T : V \to W$ be a linear transformation. If V is ____________________, then

________________________________.

Examples

(i) $T_o : V \to W$ defined by $T_o(x) = 0$.

$R(T_o) =$ ____________________; $N(T_o) =$ ____________________.

(ii) $T_1 : \mathcal{P}_n(F) \to \mathcal{P}_{n+2}(F)$ defined by multiplying by $T_1(p(x)) = x^2p(x)$.

$R(T_1) =$ ____________________; $N(T_1) =$ ____________________.

(iii) $T_2 : \mathcal{P}_n(F) \to \mathcal{P}(F)$ defined by differentiation: $T_2(x) = p'(x)$.

$R(T_2) =$ ____________________; $N(T_2) =$ ____________________.

(iv) $T_3 : \mathbb{R}^\infty \to \mathbb{R}^\infty$ defined by $T_3((x_1, x_2, x_3...)) = (x_2, x_3, ...)$

$R(T_3) =$ ____________________; $N(T_3) =$ ____________________.

Matrix Representations of a Linear Transformation

Given a linear transformation $T : V \to W$, where V and W are finite-dimensional vector spaces, we can write a *representation* of T as a matrix. It is important to understand that there are many, many matrix representations for each linear transformation – we must fix a basis for V and for W before writing down the representation!

Fix an ordered basis $\beta = \{u_1, u_2, ..., u_n\}$ of an n-dimensional vector space V. Then, for $x \in V$, we know we can write $x = a_1u_1 + a_2u_2 + a_nu_n$ where $a_1, ..., a_n$ are *unique* scalars. In other words, given that we know the basis β, the numbers $a_1, a_2, ..., a_n$ uniquely determine x! We denote the **coordinate vector of** x **relative to** β by

$$[x]_\beta = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

Notice that $[x]_\beta$ is a vector in the space F^n. Also notice that $[u_i]_\beta = e_i$; in other words, the linear map that takes $x \to [x]_\beta$ maps the given basis vectors to the standard basis vectors for F^n.

Fix two finite-dimensional vector spaces V and W and a linear transformation $T : V \to W$. Given an ordered basis $\beta = \{v_1, v_2, ..., v_n\}$ for V and an ordered basis $\gamma = \{w_1, w_2, ..., w_m\}$ for W, we can define the **matrix representation of T in the ordered bases β and γ**, denoted by $A = [T]_\beta^\gamma$, whose entries are the *unique* scalars $a_{ij} \in F$ such that $T(v_j) = \sum_{i=1}^{m} a_{ij} w_i$

We have just seen how, if $\dim(V) = n$ and $\dim(W) = m$, a linear transformation $T \in \mathcal{L}(V, W)$ can be represented as an $m \times n$ matrix; it is also true that every $m \times n$ matrix defines a linear transformation from V to W. It turns out that the space of *all* linear transformations from V to W, denoted by $\mathcal{L}(V, W)$ is a vector space that is, in some sense, the"same" as $M_{m\times n}(F)$!

THEOREM The composition of two linear transformations is a linear transformation.

THEOREM Consider finite-dimensional vector spaces V, W, and Z and linear transformations $T : V \to W$ and $U : W \to Z$. Given ordered bases α, β, and γ for V, W, and Z,

$$[UT]_{\_\_}^{\_\_} =$$

Invertibility and Isomorphism

DEFINITION Let $T : V \to W$ be a linear transformation. T is **invertible** if there exists an **inverse** $U : W \to V$, that is, a function with the property $TU =$ ____ and $UT =$ ____. *Note: If T is invertible, the inverse is denoted T^{-1} and is unique! Also, since T is linear, T^{-1}, if it exists, must also be linear.*

Recall the definition of the inverse of a matrix. The following theorem states that the inverse of the matrix representation of a linear transformation is related in the obvious way to the inverse of the linear transformation itself!

THEOREM Let V and W be finite-dimensional vector spaces with ordered bases β and γ. A linear transformation $T : V \to W$ is invertible if and only if its matrix representation $[T]_\beta^\gamma$ is invertible. Moreover, the matrix representation of T^{-1} is the inverse of the matrix representation of T: that is, ________________.

DEFINITION Let V and W be vector spaces. V is **isomorphic** to W if there exists $T \in \mathcal{L}(V, W)$ such that T is invertible.

THEOREM Let V and W be finite-dimensional vector spaces. V is isomorphic to W if and only if $\dim(V) = \dim(W)$.

THEOREM Let V and W be finite-dimenensional vector spaces over F with $\dim(V) = m$ and $\dim(W) = n$. The spaces $\mathcal{L}(V, W)$ and $M_{m\times n}(F)$ are isomorphic.

Proof: Fix ordered bases β and γ of V and W, respectively. To complete the proof, show that the linear function $\Phi : \mathcal{L}(V, W) \to M_{m\times n}(F)$ defined by $\Phi(T) = [T]_\beta^\gamma$ is invertible.