Math 108b: Solutions 1/7/11

A matrix representation

Let $S : \mathcal{P}_2 \to \mathcal{P}_1$ be the linear transformation given by $S(p(x)) = p'(x)$ for every polynomial $p(x) \in \mathcal{P}_2$. Fix two ordered bases: $\beta = \{1, x, x^2\}$ for the domain $\mathcal{P}_2$ and $\gamma = \{x+1, x-1\}$ for the domain $\mathcal{P}_1$. (Question: how can you check that γ is indeed a basis?)

We want to find the matrix representation of S with respect to these two bases: I.e., find $[S]_\beta^\gamma$.

Consider what happens to each basis vector – 1, x, and x^2 – under the transformation S. Then, make sure to write the result *in terms of the basis* γ! (That is, as a linear combination of the polynomials $x+1$ and $x-1$.)

$$\begin{aligned} S(1) &= 0 = 0 \cdot (x+1) + 0 \cdot (x-1) \\ S(x) &= 1 = \tfrac{1}{2} \cdot (x+1) - \tfrac{1}{2} \cdot (x-1) \\ S(x^2) &= 2x = 1 \cdot (x+1) + 1 \cdot (x-1) \end{aligned}$$

The 2×3 matrix $[S]_\beta^\gamma$ is therefore

$$[S]_\beta^\gamma = \begin{pmatrix} 0 & \frac{1}{2} & 1 \\ 0 & -\frac{1}{2} & 1 \end{pmatrix}$$

To see why this matrix represents the transformation S, think about any polynomial $p(x) \in \mathcal{P}_2$. We can write $p(x)$ in terms of the the basis β: $p(x) = a + bx + cx^2$, for unique scalars a, b, and c. The transformation S applied to $p(x)$ then gives $S(p(x)) = b + 2cx$. But, to write this result in terms of the basis γ, solve the following equation for α_1 and α_2:

$$S(p(x)) = b + 2cx = \alpha_1(x+1) + \alpha_2(x-1)$$

You should find that $\alpha_1 = c + \frac{b}{2}$ and $\alpha_2 = c - \frac{b}{2}$. Therefore, associating the polynomial $p(x)$ with the (column) vector $[p(x)]_\beta = (a, b, c)^T$, we have the right result:

$$[S]_\beta^\gamma [p(x)]_\beta = \begin{pmatrix} 0 & \frac{1}{2} & 1 \\ 0 & -\frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} c + \frac{b}{2} \\ c - \frac{b}{2} \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}.$$

The answer is exactly the vector associated with the polynomial $S(p(x)) = b + 2cx \in \mathcal{P}_1$, when written in terms of the basis γ.

Compositions

Define the linear transformations $T : \mathcal{P}_n \to \mathcal{P}_{n+2}$ by $T(p(x)) = x^2p(x)$ and $S : \mathcal{P}_{n+2} \to \mathcal{P}_{n+1}$ by $S(p(x)) = p'(x)$.

Then, the composition $ST : \mathcal{P}_n \to \mathcal{P}_{n+1}$ is also a linear transformation. It is defined by

$$ST(p(x)) = S(T(p(x))) = S(x^2p(x)) = (x^2p(x))' = 2xp(x) + x^2p'(x).$$

For each of the spaces $\mathcal{P}_m$, fix the standard basis $\beta = \{1, x, x^2, ..., x^m\}$. The $(n+2)\times(n+1)$ matrix representation of ST is then (check what happens to each basis vector!)

$$[ST] = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 2 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 3 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 4 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \text{n -1} & 0 \\ 0 & 0 & 0 & \cdots & 0 & \text{n+2} \end{pmatrix}$$

Notice you can find the matrix representation of the composition by multiplying the matrices $[S]$ and $[T]$ together! (We found these matrices in class on Wednesday; notice the size of $[S]$ is $(n+2)\times(n+3)$ and the size of $[T]$ is $(n+3)\times(n+1)$.)

$$[S][T] = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \text{n +1} & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & \text{n+2} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 2 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 3 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 4 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \text{n -1} & 0 \\ 0 & 0 & 0 & \cdots & 0 & \text{n+2} \end{pmatrix} = [ST]$$

You can do the same computation for the composition TS (first differentiate, then multiply by x^2). Notice $TS(p(x)) = x^2p'(x)$ is a different transformation than ST!