Theorem. (The Archimedean Property of $\mathbb{R}$) The set $\mathbb{N}$ of natural numbers is unbounded above in $\mathbb{R}$.

Note: We will use the completeness axiom to prove this theorem. Although the Archimedean property of $\mathbb{R}$ is a consequence of the completeness axiom, it is weaker than completeness. Notice that $\mathbb{N}$ is also unbounded above in $\mathbb{Q}$, even though $\mathbb{Q}$ is not complete. We also have an example of an ordered field for which the Archimedean property does not hold! $\mathbb{N}$ is bounded above in $\mathbb{F}$, the field of rational polynomials!

Proof by contradiction. If $\mathbb{N}$ were bounded above in $\mathbb{R}$, then by ________________ ____________ $\mathbb{N}$ would have a ________________. I.e., there exists $m \in ____$ such that $m = ______________$. Since $m$ is the ________________, _____ is not an upper bound for $\mathbb{N}$. Thus there exists an $n_o \in \mathbb{N}$ such that $n_o > ______$. But then $n_o + 1 > ______$, and since $n_o + 1 \in \mathbb{N}$, this contradicts _______ _________________.

The Archimedean property is equivalent to many other statements about $\mathbb{R}$ and $\mathbb{N}$.

12.10 Theorem. Each of the following is equivalent to the Archimedean property.

(a) For every $z \in \mathbb{R}$, there exists an $n \in \mathbb{N}$ such that $n > z$.

(b) For every $x > 0$ and for every $y \in \mathbb{R}$, there exists an $n \in \mathbb{N}$ such that $nx > y$.

(c) For every $x > 0$, there exists an $n \in \mathbb{N}$ such that $0 < \frac{1}{n} < x$.

The proof is given in the book. The idea is that (a) is the same as the Archimedean property because (a) is essentially the statement that “For every $z \in \mathbb{R}$, $z$ is not an upper bound for $\mathbb{N}$.” Then, it is fairly easy to see why (b) and (c) follow.
**Theorem (Q is dense in \( \mathbb{R} \)).** For every \( x, y \in \mathbb{R} \) such that \( x < y \), there exists a rational number \( r \) such that \( x < r < y \).

Notes: The idea of this proof is to find the numerator and denominator of the rational number that will be between a given \( x \) and \( y \). To do this, we first find a natural number \( n \) for which \( nx \) and \( ny \) will be more than one unit apart (this will require the Archimedean property!) Notice that the closer together \( x \) and \( y \) are, the bigger this \( n \) will need to be! Picture (assuming \( x > 0 \)):

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0 \quad x \quad y \quad nx \quad ny
\frac{m}{n}
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Since \( nx \) and \( ny \) are far enough apart, we expect that there exists a natural number \( m \) in between \( nx \) and \( ny \). Finally, \( \frac{m}{n} \) will be the rational number in between \( x \) and \( y \)!

**Proof.** Let \( x, y \in \mathbb{R} \) such that \( x < y \) be given. We will first prove the theorem in the case \( x > 0 \). Since \( y - x > 0 \), \( \frac{y}{x} \in \mathbb{R} \). Then, by the Archimedean property, there exists an \( n \in \mathbb{N} \) such that \( n > \frac{y}{x} \). Therefore, \( \frac{y}{x} < ny \). Since we are in the case \( x > 0 \), \( \frac{y}{x} > 0 \) and there exists \( m \in \mathbb{N} \) such that \( m - 1 \leq \frac{y}{x} < m \) (The proof that such an \( m \) exists uses the well-ordering property of \( \mathbb{N} \); see Exercise 12.9.) Then, \( ny > \frac{y}{x} \geq \frac{y}{x} \). Thus \( nx < m < ny \). It then follows that the rational number \( r = \frac{m}{n} \) satisfies \( x < r < y \).

Now, in the case \( x \leq 0 \), there exists \( k \in \mathbb{N} \) such that \( k > |x| \). Since \( k - |x| = k + x \) is positive and \( k + x < k + y \), the above argument proves that there is a rational number \( r \) such that \( k + x < r < k + y \). Then, letting \( r' = r - k, r' \) is a rational number such that \( x < r' < y \). □

It is also true that for every \( x, y \in \mathbb{R} \) such that \( x < y \), there exists an *irrational* number \( w \) such that \( x < w < y \). Combining these facts, it follows that for every \( x, y \in \mathbb{R} \) such that \( x < y \) there are in fact infinitely many rational numbers and infinitely many irrational numbers in between \( x \) and \( y \)!