Math 117: Final Review Problems

1. Prove or give a counterexample.

(a) The union of every collection of open sets is open.

(b) The intersection of every collection of open sets is open.

2. Give an example of nonempty closed sets $C_1 \supset C_2 \supset C_3 \supset \ldots$ such that $\bigcap_{n=1}^{\infty} C_n$ is empty.

3. (a) State the definitions of maximum and supremum.

(b) Let $S \subseteq \mathbb{R}$ be a nonempty compact set. Prove that S has a maximum.

4. (a) Let $S \subseteq \mathbb{R}$. Define the sets $\operatorname{cl} S$ and $\operatorname{int} S$ (and any words you use in the definitions, other than ϵ-neighborhoods.)

(b) Let $S, T \subseteq \mathbb{R}$. Prove that $\operatorname{int} S \cup \operatorname{int} T \subseteq \operatorname{int}(S \cup T)$.

(c) Prove that if $S \subseteq \mathbb{R}$ is open, then $S \subseteq \operatorname{int}(\operatorname{cl} S)$.

(d) Give a counterexample to show that the statement "If $S \subseteq \mathbb{R}$ is open, then $S = \operatorname{int}(\operatorname{cl} S)$" is false.

5. (a) State the Bolzano-Weierstrass theorem.

(b) Let $S \subseteq \mathbb{R}$. Prove that S is compact iff every infinite subset of S has an accumulation point in S. (Hint: for one direction, do a proof by contradiction!)

6. (a) State the definition of a compact set.

(b) Prove that $\left\{\frac{1}{n^2} : n \in \mathbb{N}\right\}$ is not compact, using *only* the definition of compact.

7. (a) Consider the sequence defined by $s_n = \frac{\sqrt{n}}{n^2-3}$. Prove that the sequence (s_n) converges to 0 using only the definition of convergence.

(b) Consider the sequence defined by $s_n = \frac{n^2}{5n+1}$. Prove that the sequence (s_n) diverges to $+\infty$.

8. (a) State the definition of a Cauchy sequence.

(b) Prove that every convergent sequence is Cauchy.

9. Without quoting any theorems from the book, prove the following.
(a) Suppose (t_n) converges to t. If $t_n \geq 0$ for all $n \in \mathbb{N}$, then $t \geq 0$.

(b) Let (a_n) and (b_n) be two sequences such that $\lim a_n = 0$ and such that for some $m \in \mathbb{R}$ and $k > 0$, $0 \leq b_n \leq k a_n$ for all $n \geq m$. Prove that $\lim b_n = 0$

10. (a) State the Monotone Convergence Theorem

(b) Prove that the sequence defined by $s_1 = 2$ and $s_{n+1} = \sqrt{2s_n + 1}$ for all $n \in \mathbb{N}$ is convergent and find its limit.