Math 117: The Completeness Axiom

Theorem. Let D be a natural number such that D is not a perfect square. There is no rational number whose square equals D. (I.e., $\sqrt{D}$ is not a rational number.)

Lemma. Let D be a natural number such that D is not a perfect square. Then there exists a natural number λ such that $\lambda^2 < D < (\lambda + 1)^2$.

Proof of lemma. Homework!

Proof of theorem by contradiction. Assume r is a rational number such that $r^2 = D$. Obviously, $r \neq$ ____ since $D \geq 1$. We can assume $r > 0$ (otherwise, ____ is a rational number such that (____)$^2 = r^2 = D$ and ____ > 0.) Since r is rational and $r > 0$, there exist positive integers u and t with ______________________ such that $r = \frac{t}{u}$. Then, $t^2 =$ ______. Using the lemma, there exists a natural number λ such that $\lambda^2 < D < (\lambda + 1)^2$. Therefore,

$$\underline{\qquad\qquad\qquad\qquad} < \underline{\qquad\qquad} = t^2 < \underline{\qquad\qquad\qquad\qquad}.$$

Therefore, since u, t, and λ are positive, ______ $< t <$ ______. These inequalities tell us that __________ is positive and that __________ $< u$. We rewrite the fraction $\frac{t}{u}$ as follows:

$$\frac{t}{u} = \frac{t\,(\qquad\qquad\qquad\qquad)}{u\,(\qquad\qquad\qquad\qquad)} = \frac{\frac{t}{u}(\qquad\qquad\qquad\qquad)}{(\qquad\qquad\qquad\qquad)} = \frac{(\qquad\qquad\qquad\qquad)}{(\qquad\qquad\qquad\qquad)}.$$

Letting $t' =$ __________ and $u' =$ __________, notice these are both positive integers and $r =$ ______. Since $u' < u$ and $t' =$ __________ $< t$, this contradicts the fact that $r = \frac{t}{u}$ was ______________________________________. □

Note. Theorem 12.1 in the book is stated only for prime natural numbers. However, the proof can be adapted to work for all natural numbers that are not a perfect squares by using a little bit of number theory (like prime factorizations). Notice the proof given in the book is also a proof by contradiction and even arrives at the same contradiction we did (after you assume the rational number such that $r^2 = D$ is written in lowest terms, it turns out it couldn't have been!)

Consider the set $T = \{r \in \mathbb{Q} : 0 < r^2 < 2\}$.

Does this set have an *upper bound* in $\mathbb{Q}$?

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But we don't expect it to have a "least upper bound" (a *supremum*) in $\mathbb{Q}$. However, we do expect T to have a *supremum* in $\mathbb{R}$ – namely, we expect $\sqrt{2}$ to be the "least upper bound."

Definitions. Let S be a subset of $\mathbb{R}$.

- A real number x is an *upper bound* for S iff ____________________ for every ______________.
- A real number s is the *supremum* of S ($s = \sup S$) iff both
  - (a) s is __________________________________ for S.
  - (b) for every x _________, there exists k _________ such that _____________.
- A real number m is the *maximum* of S iff m is _____________________ and __________.

We can similarly define *lower bound*, *infimum* (the "greatest lower bound"), and *minimum*. (Homework: Read Practice 12.6)

The Completeness Axiom. For every nonempty subset S of the real numbers is that is bounded above, $\sup S$ exists and is a real number.

Using the completeness axiom we can prove that $\sqrt{2}$ exists! In other words, there exists a positive number $x \in \mathbb{R}$ such that $x^2 = 2$. In fact, we will prove that $\sqrt{D}$ exists for every natural number D.

Theorem. Let D be a natural number. Then, there exists a positive real number x such that $x^2 = D$.

Proof. Let $S = \{r \in \mathbb{Q} : 0 < r^2 < D\}$. Since $D \geq 1$, ____ $\in S$ and S is nonempty. Also, S is bounded above by ____ since for every $r \in S$, ____________________ and therefore, ______. Therefore, by the completeness axiom, there exists $x \in$ ____ such that __________. Notice that x is positive since x is ___________________________ and $1 \in S$. We plan to show that $x^2 = D$ by contradiction.

Suppose $x < D$.

> Prove that this assumption leads to a contradiction (on another sheet of paper).
> Hint: What property of x will be impossible if it is the case that $x < D$? This is the fact that you should try to contradict!

Suppose $x > D$.

> Prove that this assumption also leads to a contradiction.
> Hint: In this case, what do you know about x that you will be trying to contradict?

Since both $x < D$ and $x > D$ lead to a contradiction, we must have that $x = D$. □