Math 117: Midterm Review Problems

1. Prove the following, using the definition of the interior of a set:
Given sets $S, T \subseteq \mathbb{R}$, $\text{int}(S \cap T) = \text{int } S \cap \text{int } T$.

2. (a) State the theorem on the density of $\mathbb{Q}$ in $\mathbb{R}$.
(b) Let $x \in \mathbb{R}$. Prove that $x = \sup\{q \in \mathbb{Q} : q < x\}$. (*Hints: Prove that the set* $S = \sup\{q \in \mathbb{Q} : q < x\}$ *has a supremum. Let* $y = \sup S$. *Then, show that* $y \neq x$ *leads to contradiction.*)

3. (a) Write the definition of the infimum of a set $S \subseteq \mathbb{R}$. (Define any words you use in your definition!)
(b) Prove that the infimum of a set $S \subseteq \mathbb{R}$ is unique.

4. (a) State the Well-Ordering Property of $\mathbb{N}$.
(b)Fill in the blanks of the following proof.
Theorem: The well-ordering property of $\mathbb{N}$ implies the principle of mathematical induction.
Proof: (by contradiction) Suppose that there exists a statement $P(n)$ such that
(a) $P(1)$ is true
(b) For all $k \in \mathbb{N}$, ____
(c) and ____
Define the set

$$S = \{n \in \mathbb{N}, \underline{\hspace{6cm}}\}$$

Then, by hypothesis (c) $S \subseteq \mathbb{N}$ is non-empty, so there exists $m \in$ _____ such that ______________________________ by the ____. Since $P(1)$ is true by hypothesis (a), _____$\notin S$. Hence, $m >$_____. It follows that $m - 1$ is a natural number, and since $m - 1 < m$, _____________. By the definition of S, $P(m-1)$ is _____________. Then, (b) implies that _______________________________________. This implies that ___________________, which is a contradiction.

5. Prove that if $S \subseteq \mathbb{R}$ has a minimum, then $\text{bd } S \cap S \neq \emptyset$.

6. (a)Write definitions (in terms of neighborhoods!) for an accumulation point, an isolated point, and a boundary point of a set S.
(b) Prove that a boundary point of a set $S \subseteq \mathbb{R}$ is either an accumulation point of S or an isolated point of S.

7. Prove the statement if it is true (but do not quote any theorems from the book in your proof); otherwise, give a counterexample.
(a) If $x \geq y$ and $x \leq y + \epsilon$ for all $\epsilon > 0$, then $x = y$.
(b) If $x > y - \epsilon$ for all $\epsilon > 0$, then $x > y$.

8. (a) Write the definition of a compact set.
(b) Prove, using the definition of a compact set, that $(0, 1]$ is not compact.