Math 117: Sequences, Part II

Example 1. Show that $\lim_{n\to\infty}\left(1+\frac{(-1)^n}{n}\right)=1$.

Example 2. Show that $\lim_{n\to\infty}\frac{4n^3-1}{2n^3+3}=$ ____.

Example 3. Show that $\frac{4n^2+7}{2n^4-85}$ converges (using the definition of convergence).

Ideas. We want to show that $\lim_{n\to\infty}\frac{4n^2+7}{2n^4-85}=$ ____. If we let ______________________, our goal is to prove that

$$\left|\frac{4n^2+7}{2n^4-85}\right| < \text{____}.$$

whenever ______________________. Our idea is to first simplify the fraction by showing that essentially, $\frac{4n^2+7}{2n^4-85}\approx$ *const* ____ for large n. It would be enough to show that, if

______________________,

$$\left|\text{______________}\right| \leq k_1 \text{ ________}$$

$$\left|\text{______________}\right| \geq k_2 \text{ ________}$$

for $k_1>0$ and $k_2>0$, because then, we would have that $\frac{|4n^2+7|}{|2n^4-85|}\leq$ ____________. We expect to be able to do this with, for example $k_1=$ ____ and $k_2=$ ____.

Scratch work:

Proof. We will show that $\lim_{n\to\infty} \dfrac{4n^2+7}{2n^4-85} = \_\_\_\_\_$. Let _________________________. Let $m = \_\_\_\_\_$. If $n \geq \_\_\_\_\_$, then both $|\_\_\_\_\_\_\_\_\_|$ and $|\_\_\_\_\_\_\_\_\_|$ are positive, so

$$\left|\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\right| = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ < \_\_\_\_\_\_\_\_\_ \quad \text{because } \_\_\_\_\_\_\_\_\_\_\_\_\_$$

$$\left|\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\right| = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ \geq \_\_\_\_\_\_\_\_\_ \quad \text{because } \_\_\_\_\_\_\_\_\_\_\_\_\_$$

Therefore,

$$\left|\frac{4n^2+7}{2n^4-85}\right| = \frac{|4n^2+7|}{|2n^4-85|} = \frac{4n^2+7}{2n^4-85} \quad \text{because } \_\_\_\_\_\_\_\_\_\_\_\_\_$$

$$\leq \_\_\_\_\_\_\_\_\_ = \_\_\_\_\_\_\_\_\_ \quad \text{because } \_\_\_\_\_\_\_\_\_\_\_\_\_$$

Let $N = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$. Then, if $n > \_\_\_\_\_$,

$$\left|\frac{4n^2+7}{2n^4-85}\right| \leq \_\_\_\_\_\_ < \_\_\_\_\_\_ \leq \epsilon.$$

Therefore, for every $\epsilon > 0$, we have found that there exists _____ such that _______________ __________________________. This is the definition of $\lim_{n\to\infty} \dfrac{4n^2+7}{2n^4-85} = \_\_\_\_\_$. □

Theorem. Let (s_n) and (a_n) be sequences of real numbers and let $s \in \mathbb{R}$. Assume $\lim a_n = 0$. Also, assume there exists $k > 0$ and $m \in \mathbb{N}$ such that

$$|s_n - s| \leq k|a_n| \text{ for all } n \geq m.$$

It follows that $\lim s_n = s$.

Proof. _________________________. Since $\lim a_n = 0$, there exists $N_1 \in \mathbb{R}$ such that for every _________, _________________________. Let $N = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$. Then for every _________, both _________ and _________. So, by assumption and by the definition of N_1,

$$|s_n - s| \leq \_\_\_\_\_\_\_\_\_\_ < \_\_\_\_\_\_\_\_\_\_ = \_\_\_\_\_\_.$$

Therefore, $\lim s_n = s$. □

Example 4. Prove that the sequence (s_n) where $s_n = \dfrac{4n^2+7}{2n^4-85}$ converges using this theorem.

Let $m =$ _____. We know that for $n \geq m$,

$$\left|\frac{4n^2+7}{2n^4-85}\right| = \underline{\hspace{10cm}}$$
$$\leq$$

(See the calculations in Example 3.) Therefore, $|s_n - 0| = |s_n| \leq k|a_n|$ where $k =$ _____ $>$ 0 and $a_n =$ _____. Since _____ $\rightarrow$ _____, the above theorem implies that s_n converges ($\lim s_n =$ _____).

Example 5. Prove that $\lim n^{\frac{1}{n}} = 1$. (See Example 16.11 in the book.)

Example 6. Show that $\left((-1)^n\right)$ *diverges.*

We show this by contradiction. Assume the sequence $\left((-1)^n\right)$ converges to a limit s. Then, let $\epsilon_1 = 1$.

According the definition of convergence, __________________________________,

$$\left|(-1)^n - s\right| < \underline{\hspace{1.5cm}}.$$

Since there exists $n_1 > N$ such that n_1 is _______, we must have that |_____________| < 1. Also, there exists $n_2 > N$ such that n_2 is _______, so |_____________| < 1. Therefore, we have both ____ $< s <$ ____ and ____ $< s <$ ____. Since this is a contradiction, the sequence (s_n) is not convergent.

Theorem. Every convergent sequence is bounded.

Proof. Let (s_n) be a convergent sequence. Let $\lim s_n = s$. From the definition of convergence, we know that there exists $N \in \mathbb{R}$ such that __________________________ whenever __________. But, then if __________,

$$\underline{\hspace{5cm}} \leq |s_n - s| < \underline{\hspace{1.5cm}}.$$

This implies ________________. Let $M =$ _____________________________________. Then, we have that $|s_n| \leq M$ for all $n \in \mathbb{N}$, so (s_n) is bounded. $\square$