Math 117: Sequences, Part III

Limit Theorems

Given two sequences (s_n) and (t_n), we can define new sequences such as (a_n) where $a_n = s_n + t_n$ and (b_n) where $b_n = s_n t_n$. If we know that (s_n) and (t_n) are convergent sequences we can prove that (a_n) and (b_n) are also convergent.

Theorem. Suppose that (s_n) and (t_n) are convergent sequences with $\lim s_n = s$ and $\lim t_n = t$, and suppose that $k \in \mathbb{R}$. Then, the following sequences also converge: $(s_n + t_n)$; (ks_n); $(k + s_n)$; $(s_n t_n)$; (s_n/t_n), provided that __________ for all n and __________. Moreover,

- **(a)** $\lim(s_n + t_n) =$ __________
- **(b)** $\lim(ks_n) =$ __________
- **(c)** $\lim(k + s_n) =$ __________
- **(d)** $\lim(s_n t_n) =$ __________
- **(e)** $\lim(s_n/t_n) =$ __________, provided that __________ for all n and __________.

Proof of (c). ((a) – (e) are proven in the book or are exercises.) We want to show: For all $\epsilon > 0$, there exists $N \in \mathbb{R}$ such that _________________________ whenever __________.

Let $\epsilon > 0$ be given. Consider |____________________|. By the triangle inequality,

$$\left| \underline{\hspace{4cm}} \right| =$$

$$\leq |s_n||t_n - t| + |s_n - s||t|.$$

Since (s_n) converges, we know that (s_n) is _______________. Therefore, there exists $M > 0$ such that ____________________ for all n. Since $\epsilon > 0$ and $M > 0$, $\frac{\epsilon}{2M} > 0$. Using the definition of convergence, we know there exists $N_1 \in \mathbb{R}$ such that _________________________ whenever __________. Also, since $\frac{\epsilon}{2(|t|+1)} > 0$ and since we know that $s_n \to s$, we can use the definition of convergence to say there exists $N_2 \in \mathbb{R}$ such that _________________________ whenver __________. Let $N =$ ____________________. Then, if $n > N$,

$$\left| \underline{\hspace{4cm}} \right| \leq |s_n||t_n - t| + |s_n - s||t| \leq \underline{\hspace{1cm}}\,|t_n - t| + \underline{\hspace{1cm}}\,|s_n - s|$$

$$< \underline{\hspace{6cm}} \leq \underline{\hspace{2.5cm}} = \epsilon.$$

Hence, $\lim(s_n t_n) =$ __________. □

Example. Show that $\lim_{n\to\infty} \dfrac{4n^2+7}{2n^4-85} = 0.$

We can rewrite $\dfrac{4n^2+7}{2n^4-85} =$

Since $\lim$ ____________ $=$

Similarly, $\lim$ ____________ $=$

By **(e)**, $\lim \dfrac{4n^2+7}{2n^4-85} =$ $\square$

Theorem. Suppose that (s_n) and (t_n) are convergent sequences with $\lim s_n = s$ and $\lim t_n = t$. If $s_n \leq t_n$ for all n, then $s \leq t$.

Proof by contradiction. Assume that both ____________________ and _______________. Then, let $\epsilon =$ __________. Since __________, $\epsilon > 0$, so applying the definition of convergence, there exist $N_1 \in \mathbb{R}$ and $N_2 \in \mathbb{R}$ such that

$$\_\_\_\_\_\_\_\_\_\_ < s_n < \_\_\_\_\_\_\_\_\_\_$$

$$\_\_\_\_\_\_\_\_\_\_ < t_n < \_\_\_\_\_\_\_\_\_\_$$

Let $N =$ _______________. Then, if $n > N$, we have that

$$t_n < \_\_\_\_\_\_\_\_\_\_ = \_\_\_\_\_\_\_\_\_\_ < s_n.$$

This contradicts the assumption that ___________________. Thus, $s \leq t$. $\square$

Ratio Test

Theorem: Ratio Test Suppose that (s_n) is a sequence of positive terms (i.e., $s_n \geq 0$ for all n) and that the sequence of ratios converges to L (i.e., $\lim s_{n+1}/s_n = L$). If $L < 1$, then $\lim s_n = 0$.

Notes. We expect this to work because, if $s_{n+1}/s_n \to L$, then for large n, $s_{n+1}/s_n \approx L$. Therefore, for large n, we expect to have $s_{n+1} \approx Ls_n \approx L(Ls_{n-1}) = L^2 s_{n-1} \approx \ldots \approx L^{n-1}s_1$. But, if $L < 1$, then $L^{n-1}s_1 \to 0$.

Example. Show that $\lim \dfrac{n^p}{n!} = 0$ for all $p > 0$. Let $s_n =$ ________. We want to show that the sequence (s_n) converges to ______. Notice that every s_n is ________________. Consider the sequence of ratios $t_n = \dfrac{s_{n+1}}{s_n}$:

$$|t_n| =$$

We will show that $\lim t_n =$ ______. For all $n \in \mathbb{N}$, $(n+1) \leq$ ________. Therefore,

$$|t_n| =$$

Hence, for all $n \geq 1$, $|t_n| \leq$ _______$|a_n|$, where $a_n =$ ___________ is a sequence that converges to ______. Therefore, $t_n \to$ ______. Since the sequence of ratios $\dfrac{s_{n+1}}{s_n} \to$ ______ < 1, the sequence (s_n) must converge to ______ by the ____________________________.

Proof of the Ratio Test.

Infinite Limits

Definitions. A sequence (s_n) **diverges to** $+\infty$ iff

for every __________, there exists __________ such that for all __________, _______________.

A sequence (s_n) **diverges to** $-\infty$ iff

for every __________, there exists __________ such that for all __________, _______________.

If (s_n) diverges to $\pm\infty$, we use the notation $\lim s_n = \pm\infty$. (*This does not mean that the limit of* (s_n) *exists or that* (s_n) *converges; it just indicates that* (s_n) *diverges in a special way.*)

Example. Prove that the sequence (s_n) where $s_n = \dfrac{4n^2}{2n-5}$ diverges to $+\infty$.

Given ______, the idea is to show that for ______________________, $\dfrac{4n^2+1}{2n+5} \geq$. For all n,

we have that $4n^2 + 1 \geq$ _______. Also, for all n, we have that $2n + 5 \leq$ _______________ $=$ _______.

Given ______, let _______________. Then, if __________,

$$\frac{4n^2+1}{2n+5} \geq$$

Therefore $\lim \dfrac{4n^2+1}{2n+5} = +\infty$.

Example. Show that $\dfrac{n!}{n^p}$ diverges. (*Hint: Use Theorem 17.13.*)