Math 117: Open and Closed Sets of $\mathbb{R}$

To do analysis, we want to make precise what what we mean by "close." We will start with the idea that one way to measure how close two real numbers are is to talk about the sets that contain them both. To this end, we define "ϵ-neighborhoods" (balls of radius ϵ).

Definition Let $x \in \mathbb{R}$ and let $\epsilon > 0$.

· The ϵ-*neighborhood of* x is defined to be the set $N(x, \epsilon) =$ ________________________.

· The *deleted* ϵ-*neighborhood of* x is the set $N^*(x, \epsilon)$ ________________________.
(Notice that $N^*(x, \epsilon) =$ __________________.)

We can define ϵ-neighborhoods for any set X (instead of $\mathbb{R}$), as long as we have a way to measure distance in X! The absolute value in $\mathbb{R}$ measures the distance from 0, and $|x - y|$ measures the distance between x and y, so in $\mathbb{R}$ the (usual) distance between x and y is $d(x, y) = |x - y|$. For a set X with a distance d, if $x \in X$ and $\epsilon > 0$, $N(x, \epsilon) = \{y \in X : d(x, y) < \epsilon\}$. You have probably seen many sets on which you can define a distance in other classes: some examples are $\mathbb{R}^2$, $\mathbb{C}$, and the set of 3×3 matrices.

Our goal is to define what it means for subsets of $\mathbb{R}$ to be *open* or *closed*. First, we will need to define two different kinds of points in a set: *interior* and *boundary* points.

Definitions Let S be a subset of $\mathbb{R}$.

· $x \in S$ is an *interior point* of S iff there exists $\epsilon > 0$ such that __________________.

· $x \in S$ is a *boundary point* of S iff for every $\epsilon > 0$, __________________ and __________________.

· The set of all interior points of S is denoted $\operatorname{int} S$, and the set of all boundary points of S is denoted $\operatorname{bd} S$.

Examples:

(a) $S_a = [0, 3) \cup [3, 5) \cup \{6\}$

$\operatorname{bd} S_a =$ ________________________ $\operatorname{int} S_a =$ ________________________

(b) $S_b = (0, 3) \cup (3, \infty)$

$\operatorname{bd} S_b =$ ________________________ $\operatorname{int} S_b =$ ________________________

(c) $S_c = \{r \in \mathbb{Q} : 0 < r < 1\}$

$\operatorname{bd} S_c =$ ________________________ $\operatorname{int} S_c =$ ________________________

Our definitions of open and closed sets are in terms of the boundary of the set.

Definitions Let S be a subset of $\mathbb{R}$.

· S is *closed* iff ____________________ (i.e., S contains ________ of its boundary).

· S is *open* iff ____________________ (i.e., S contains ________ of its boundary).

Notes:

- These definitions are motivated by the idea of open intervals and closed intervals. We can prove that (a, b) is open and [a, b] is closed.
- Notice that sets do not have to be either open or closed! Example: (a, b] contains only one of its boundary points so does not satisfy the definition of either open or closed.
- There are also sets that are both open and closed! These sets must contain all their boundary and none of their boundary; therefore, if a set S is both open and closed it must satisfy $\operatorname{bd} S = \emptyset$! There are only two such sets of real numbers: $\mathbb{R}$ and $\emptyset$.

Equivalent characterizations of open and closed sets:

13.7 Theorem Let S be a subset of $\mathbb{R}$

(a) S is open iff __________________.

(b) S is closed iff its complement ________ is ________.

Exercise: Use (a) to prove that the interval $(0, 1)$ is open.

13.10 Theorem

(a) The union of every collection of open sets is an open set.

(b) The intersection of every finite collection of open sets is an open set.

Proof. (a) Let a collection of open sets $\mathscr{A}$ be given. Let $S = \bigcup_{A \in \mathscr{A}} A$. If $x \in S$, then __________ for some $A \in \mathscr{A}$. Since A is open, __________. That is, there exists an $\epsilon > 0$ such that __________________. Since $A \subseteq S$, __________________ and x is an interior point of S. Therefore, S is open.

(b) [Sketch] Let $A_1, ..., A_n$, a finite collection of open sets, be given. Let $T = \bigcap_{i=1}^{n} A_i$. If $T \neq \emptyset$, let $x \in T$ and find an $\epsilon > 0$ such that $N(x, \epsilon) \subseteq T$. (What do you know about each A_i? How will you define ϵ? Why is it necessary that the collection be finite in order to define this ϵ?) If you can find such an ϵ, then you know that $x \in \operatorname{int} T$, and T is open. $\square$