Math 117: The Well-Ordering Property of $\mathbb{N}$

The well-ordering property of $\mathbb{N}$ states that "For all sets $S \subseteq \mathbb{N}$ such that $S \neq \emptyset$, there exists a least element $m \in S$ such that $m \leq t$ for all $t \in S$."

Since you will prove this by contradiction, first think about what the negation of the well-ordering property is!

Theorem. Assuming the principle of mathematical induction as an axiom, the well-ordering property of $\mathbb{N}$ holds.

Proof by contradiction. Assume the negation of the well-ordering property:

In order to use induction (and we will need strong induction – Exercise 10.25, which you have already proven follows from induction), we define the statements

$$P(n): \qquad\qquad\qquad\qquad \forall n \in \mathbb{N}.$$

Show that (a) _________ is true and that

(b) for all $k \in \mathbb{N}$, ____.

(Hints: (a) technically requires another induction proof! Therefore, first show the lemma below; you should need to refer to this lemma in the proof of (a). Then, for (b), make sure you write down what you are assuming and use contradiction to prove what you want to show.)

Lemma. For all $t \in \mathbb{N}$, $1 \leq t$.

Proof.

$\square$

Since we have shown (a) and (b), induction allows us to conclude that $P(n)$ is true for all $n \in \mathbb{N}$. This means that ______________________ for all $n \in \mathbb{N}$; therefore, ________ is empty, which is a contradiction. $\square$