Math 122B: Complex Variables

The Cauchy-Goursat Theorem

Cauchy-Goursat Theorem. If a function \( f \) is analytic at all points interior to and on a simple closed contour \( C \) (i.e., \( f \) is analytic on some simply connected domain \( D \) containing \( C \)), then

\[
\int_C f(z) \, dz = 0.
\]

Note. If we assume that \( f' \) is continuous (and therefore the partial derivatives of \( u \) and \( v \) are continuous where \( f(z) = u(x, y) + iv(x, y) \)), this result follows immediately from Green’s theorem: Letting \( R \) be the region enclosed by the curve \( C \),

\[
\int_C f(z) \, dz = \int_C (u(x, y) + iv(x, y)) \, (dx + i \, dy) = \int_C (u \, dx - v \, dy) + i \int_C (v \, dx + u \, dy)
\]

\[
= \iint_R (-v_x - u_y) \, dA + i \iint_R (u_x - v_y) \, dA = 0
\]

since \( f \) is analytic (use the Cauchy-Riemann equations!) However, the Cauchy-Goursat theorem says we don’t need to assume that \( f' \) is continuous (only that it exists!)

Theorem. (An extension of Cauchy-Goursat)
If \( f \) is analytic in a simply connected domain \( D \), then

\[
\int_C f(z) \, dz = 0
\]

for every closed contour \( C \) lying in \( D \).

Notes.

- Combining this theorem with Theorem (§42), every function \( f \) that is analytic on a simply connected domain \( D \) must have an antiderivative on the domain \( D \).

- Given two simple closed contours such that one can be continuous deformed into the other through a region where \( f \) is analytic, the contour integrals of \( f \) over these two contours have the same value! In other words, \( f \) might not be analytic in some region \( R \), but if it is analytic outside of \( R \), then the value of the contour integrals of \( f \) must be the same for all closed contours that enclose \( R \) – of course, this value doesn’t have to be 0 since \( f \) is not analytic everywhere. (See the corollary below.)
**Corollary.** Let $C_1$ be a positively oriented simple closed contour. Then, $C_1$ breaks the complex plane up into two regions: the interior of $C_1$ and the exterior of $C_1$ (by the Jordan curve theorem). Let $C_2$ be a positively oriented simple closed contour entirely inside the interior of $C_1$. If $f$ is analytic in between and on $C_1$ and $C_2$, then

$$
\int_{C_1} f(z) \, dz = \int_{C_2} f(z) \, dz.
$$

**Proof.** Connect the contours $C_1$ and $C_2$ with a line $L$ (which starts at a point $a$ on $C_1$ and ends at a point $b$ on $C_2$). Integrate over a new contour $C$ that both begins and ends at $a$: $C = (-C_2) \cup L \cup C_1 \cup (-L)$ (see the picture below - as you travel along $C$ notice that the orientation is such that the domain in between $C_1$ and $C_2$ is always to the left!) Then, since this is a closed contour, the extension of Cauchy-Goursat implies that

$$
\int_{C} f(z) \, dz = \int_{C_1} f(z) \, dz - \int_{C_2} f(z) \, dz = 0.
$$

**Example.** We can show that $\int_{C_o} \frac{1}{z} \, dz = 2\pi i$, where $C_o$ is the positively oriented circle of radius $\epsilon_o$ centered at the origin (for any $\epsilon_o > 0$).

Therefore, for any positively oriented simple closed contour $C$ whose interior contains the origin,

$$
\int_{C} \frac{1}{z} \, dz = 2\pi i.
$$

(Write out the details!)