Math 124B: PDEs

Final Review

Outline

1. Contour integration
   - See the midterm review!
   - Parameterizing contours; computing contour integrals in various ways
   - Cauchy-Goursat theorem; Cauchy's formula and inequality; maximum modulus principle
2. Taylor and Laurent series
   - A function is analytic if and only if it equals its Taylor series. What are the coefficients of the Taylor series for $f(z)$?
   - What does the radius of convergence of a Taylor series for $f(z)$ around z_o depend on?
3. (Isolated) Singularities and Residues
   - Define *removable singularity*, *pole* (of order m), and *essential singularity*. Give an example of each.
   - What is the residue at an isolated singularity? Understand different ways to compute residues at a pole.
   - Theorems
     - If $f(z)$ is analytic at z_o and $f(z_o) = 0$, then either $f(z)$ is identically 0 or $f(z)$ is not zero in the deleted neighborhood. $0 < |z - z_o| < \epsilon$ for some $\epsilon > 0$. (This implies that if a function is analytic and zero on a line, then the function is identically 0!)
     - Lemma: If f is analytic and bounded in a deleted neighborhood $0 < |z - z_o| < \epsilon$, then $f(z)$ has a removable singularity at z_o (or is analytic at z_o).
     - Know the statement of Picard's theorem (you don't need to know a proof.)
4. Real Integrals
   - See the handout from class for types of integrals and examples of how to compute them using residue theory! (Sections 71–78 in the book.)
   - You will be given *Jordan's inequality*: $\int_0^{2\pi} e^{-R\sin\theta}\, d\theta \leq \frac{\pi}{R}$.
5. Mappings
   - Linear fractional transformations
   - Other maps: z^2, $z^{\frac{1}{2}}$, e^z, and $Log(z)$ (or other branches of $log(z)$!)
   - Conformal mappings (definition only)

Problems

1. Compute the following contour integrals:
(For the function $z^{\frac{1}{2}}$, use the branch cut $-\frac{\pi}{2} < \theta < \frac{3\pi}{2}$.)

. (a) $\int_{\gamma_1} \bar{z}\, dz$ (b) $\int_{\gamma_1} z^{\frac{1}{2}}\, dz$ (c) $\int_{\gamma_2} z^{\frac{1}{2}}\, dz$

where γ_1 is the contour starting at $z_1 = 1$ and ending at $z_2 = 3 + 3i$ that is made up of the straight line between 1 and $2 + 3i$ and the straight line between $2 + 3i$ and $3 + 3i$; and γ_2 is the straight line along the $x-$ axis from -1 to $+1$.

2. Let $f(z)$ be an entire function. Prove Cauchy's inequality for the maximum modulus of $f^{(n)}(z)$ on any circle $|z - z_o| < R$ (i.e., inside $C_R(z_o)$). (Hint: Start with Cauchy's integral formula.)

3. Compute the following integrals:

. (a) $\int_{C_3(0)} \frac{5e^{2\pi iz}}{z^2 + 3z - 4}\, dz$ (b) $\int_{C_3(-1)} \frac{5z - 2}{z(z-1)}\, dz$ (c) $\int_{C_3(0)} \frac{\cos z}{(z-4)(z-2)^2}\, dz$

4. (i) State Laurent's Theorem. (Give formulas for the coefficients!)
(ii) Compute both the Laurent series of $f(z) = \frac{3}{z^2 + z}$ around the point $z_1 = 0$ and the Laurent series of f around the point $z_2 = 1$. On what domains do these series converge?

5. What is the definition of "z_o is a *zero of order* m of the function $f(z)$"? Prove that $f(z)$ has a zero of order m if and only if $f(z) = (z - z_o)^m g(z)$ for some function g that is analytic at z_o with $g(z_o) \neq 0$.

6. What does Picard's theorem tell you about the behavior of the function $e^{\frac{1}{z}}$ near $z = 0$?

7. Find the residues of $f(z)$ at each of its isolated singularities.

. (a) $f(z) = \frac{1}{e^z(z^2 + 3)}$ (b) $f(z) = \frac{\cos(z)}{e^z - 1}$ (c) $f(z) = \frac{z}{\sin(z)}$,

8. Let $g(z)$ have a pole of order one at z_o. Prove that

$$\lim_{\rho \to 0} \int_{\gamma} g(z)\, dz = \pi i \operatorname*{Res}_{z=z_o} g(z),$$

where γ is the contour $\{z(\theta) = \rho e^{i\theta} : -\pi \leq \theta \leq \pi\}$ (Hint: Consider the Laurent series of $g(z)$.)

9. Compute the following integrals: (i) $\int_0^{\infty} \frac{x \sin(ax)}{4x^2 + 5}\, dx$ (ii) P.V. $\int_{-\infty}^{\infty} \frac{1}{x^2 + 3x + 4}\, dx$

10. Compute the integral $\int_0^\infty \frac{x^a}{(x^2+2)^2}\,dx$, where $-1 < a < 3$.

11. (a) Find a linear fractional transformation that takes $z_1 = \infty$ to $w_1 = 0$; $z_2 = 1$ to $w_2 = 1$ and $z_3 = 1 + i$ to $w_3 = 2i$.

(b) For the linear fractional transformation above, what point will be mapped to ∞? What happens to the real axis $\{y = 0\}$ under this transformation?

12. (a) What happens to the domain $\{|z| \leq 1\}$ (the unit circle) under the mapping $w = z^2$? Is the mapping conformal on this domain?

(b) Where does the domain pictured below map to in the w−plane? (This domain is bounded by the circle of radius 2 and by the lines $x = 1$ and $y = 0$.) Is the mapping conformal on this domain?