Math 122B: Complex Variables

Integration

Recall the definitions of *arc*, *simple arc* and *smooth arc* (see §38). A *contour* is a continuous, piecewise smooth arc. (So any parameterization $z(t)(a \le t \le b)$ of a contour C is continous and differentiable, with $z'(t)$ piecewise continuous on $[a, b]$.) Also recall the definitions of *closed contour* and *simple closed contour*.

Definition. Given a contour C and a function $f(z)$ that is piecewise continuous along C, we define the *contour integral*

$$\int_C f(z)\,dz = \int_a^b f[z(t)]z'(t)\,dt$$

where $z(t)(a \le t \le b)$ is any parameterization of C.

Notes.

• In particular, this definition asserts that the value of $\int_a^b f[z(t)]z'(t)\,dt$ is the same for all parameterizations of the contour C.

Proof. Given two parameterizations of C, $z(t)(a \le t \le b)$ and $Z(\tau)(\alpha \le \tau \le \beta)$, there exists a one-to-one function $\phi : [\alpha, \beta] \to [a, b]$ such that $t = \phi(\tau)$ for every $\alpha \le \tau \le \beta$ (and also such that $\phi' > 0$ since the orientation stays the same!) Then, $Z(\tau) = z(\phi(\tau))$, and $Z'(\tau) = z'(\phi(\tau))\phi'(\tau)$. By making the change of variables $t = \phi(\tau)$,

$$\int_a^b f[z(t)]z'(t)\,dt = \int_\alpha^\beta f[z(\phi(\tau))]z'(\phi(\tau))\,(\phi'(\tau)\,d\tau = \int_\alpha^\beta f[Z(\tau)]Z'(\tau)\,d\tau.$$

• Consider a contour C parameterized by $z(t)(a \le t \le b)$; $-C$ is the contour described by the same set of points traversed in the opposite direction. For example, the parameterization $w(s) = z(-s)(-b \le s \le -a)$ describes the contour $-C$. Make the change of variables $s = -t$:

$$\int_C f(z)\,dz = \int_a^b f[z(t)]z'(t)\,dt = \int_{-a}^{-b} f[z(-s)]z'(-s)\,(-ds) = \int_{-a}^{-b} f[w(s)]w'(s)\,ds$$
$$= -\int_{-b}^{-a} f[w(s)]w'(s)\,ds = -\int_{-C} f(z)\,dz.$$

• The fundamental theorem of calculus holds for integrals of complex-valued functions of a real variable. If $w(t) = u(t)+iv(t)$ has an antiderivative – that is, if there exists $W(t) = U(t)+iV(t)$ such that $U'(t) = u(t)$ and $V'(t) = v(t)$ – then,

$$\int_a^b w(t)\,dt = \int_a^b u(t)\,dt + i\int_a^t v(t)\,dt = U(t)\,\Big|_a^b + iV(t)\,\Big|_a^b = U(b) - U(a) + iV(b) - iV(a)$$
$$= [U(b) + iV(b)] - [U(a) + iV(a)] = W(b) - W(a).$$

Theorem (§42). *Let $f(z)$ be continuous on a domain D.*

(a) *There exists an analytic function $F(z)$ such that $F'(z) = f(z)$ in the domain D*

if and only if

(b) $\int_C f(z)\,dz = 0$ *for all closed contours that lie entirely in the domain D*

Note. The statement (b) is also exactly the same as

(c) *For all fixed $z_1, z_2 \in D$, $\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$ for all contours C_1, C_2 (lying entirely in D) from z_1 to z_2.*

Clearly (c) implies (b) ($z_1 = z_2$ for all closed contours). For the other direction, given C_1 and C_2 starting and ending at the same points, just notice that $C_1 \cup -C_2$ is a closed contour!

Proof of Theorem (§42).

$[(a) \Rightarrow (b)]$ Assume that there exists an antiderivative $F(z)$ of the function $f(z)$.
Given a contour C inside D, let $z(t)(a \leq t \leq b)$ be a parameterization of C. Then

$$\int_C f(z)\,dz = \int_C f(z(t))z'(t)\,dt = F(z(t))\Big|_{t=a}^{b} = F(z(b)) - F(z(a)) = F(z_2) - F(z_1)$$

since the fundamental theorem of calculus holds when integrating complex-valued functions of a real variable – and $F(z(t))$ is an antiderivative for $f(z(t))z'(t)$. (Check this using the chain rule! See §38 Exercise 5.) In particular, for any closed contour C, $\int_C f(z)\,dz = 0$.

$[(b) \Rightarrow (a)]$ Assume $\int_C f(z)\,dz = 0$ for every closed contour C inside D.
Define the function $F : D \to \mathbb{C}$ by

$$F(z) := \int_0^z f(z)\,dz$$

where the integration can be taken over any contour C from 0 to z (by assumption the integral will have the same value for every such contour).

Now, we just need to prove that $F'(z) = f(z)$ for every $z \in D$! Fix $z \in D$. For any $\Delta z \neq 0$, compute

$$\frac{F(z + \Delta z) - F(z)}{\Delta z} =$$

Since $f(z) = f(z)\left[\frac{1}{\Delta z}\int \qquad dw\right] = \frac{1}{\Delta z}\int \qquad\qquad dw$, we can combine these integrals to show

$$\left|\frac{F(z + \Delta z) - F(z)}{\Delta z} - f(z)\right| = \frac{1}{|\Delta z|}\left|\int \qquad\qquad\qquad dw\right|$$

Finally, given $\epsilon > 0$, since f is continuous at $z \in D$, we know that there exists $\delta > 0$ such that

Therefore, for all Δz small enough (that is, whenever $\Delta z < \delta$),

$$\left| \frac{F(z+\Delta z) - F(z)}{\Delta z} - f(z) \right| < \frac{1}{|\Delta z|}(\qquad\qquad\qquad) = \epsilon.$$

By the definition of limit,

$$\lim_{\Delta z \to 0} \frac{F(z+\Delta z) - F(z)}{\Delta z} = f(z).$$

□

Examples. Consider C, the circle of radius 2 centered at the origin.
Let $g(z) = \frac{1}{z^2}$ and $f(z) = \frac{1}{z}$.

(1) $G(z) =$ ________ is analytic on $D_1 =$ ________ and $G'(z) =$ ________.

Therefore, $G(z)$ is an antiderivative for $g(z)$ on the domain D_1, which contains C.
By the theorem,

$$\int_C \frac{1}{z^2}\, dz = 0.$$

(2) Does $f(z)$ have an antiderivative on some domain containing C?

How can we use the theorem to compute $\int_C \frac{1}{z}\, dz$?