Math 122B: Complex Variables

Midterm Review

The midterm will cover computation of integrals and of Taylor and Laurent series. You should also be able to state all the theorems listed in italics. Any proofs on the exam will be those mentioned below in the outline or ones that are similar.

1. Integration

- Know how to parameterize contours and compute coutour integrals using the definition:

$$\int_C f(z)\,dz = \int_a^b f(z(t))\,z'(t)\,dt.$$

- Know how to use antiderivatives to compute contour integrals.
  - Examples: page 141, #2
  - Be careful when integrating a function defined using a branch cut – the function will only have an antiderivative on contours that do not touch the branch cut! See examples 3 and 4 in §43 (and the homework problem #4 on page 142).
- Know the statement of the *Cauchy-Goursat theorem* and its extension.
  - To apply either theorem, you should consider the points where the function f that you're integrating is not analytic – see exercises #1 and #2, page 153.
  - An interesting application was the homework problem #4 on page 154, in which you applied Cauchy-Goursat and also parameterized the contour to rewrite the contour integral as a sum of real integrals. Using this, you could deduce the value of a complicated real integral.
- Know the *Cauchy integral formula*.
  - This formula helps us easily calculate many contour integrals: See exercise #1, page 162.
  - The proof of this also implies that analytic functions have derivatives of all orders (you don't need to know the proof, just this fact!).
- Know *Cauchy's inequality*. (For a circle $C_R(z_o)$ on which f is analytic, Cauchy's inequality gives an upper bound for $|f^{(n)}(z_o)|$.)
  - Prove this by bounding the modulus of the contour integral that appears in Cauchy's integral formula.

- Know the statement of *Liouville's theorem*.
  - Prove Liouville's theorem using Cauchy's inequality.
  - See the homework problems #1 and #2 from page 171.
  - Prove that a polynomial $P(z)$ of degree $n \geq 1$ must have at least one zero. (Hint: Assume this is not true. Then $f(z) = 1/P(z)$ is entire. Show that $f(z)$ is bounded by showing that $|P(z)| \geq C > 0$ for z large enough.)
- Know the statement of the *maximum modulus principle*
  - Prove that if f is analytic and not constant on a domain D, then $u(x, y) = \text{Re } f(z)$ has no maximum value in D. (Hint: consider the funtion $g(z) = e^{f(z)}$.)
  - See the example on page 171, and exercises #6 and #8, page 172.

2. Taylor and Laurent Series

- Basic facts about sequences and series
- *Taylor's theorem*
  - The idea of the proof is to write down the partial sum and use Cauchy's integral formula to rewrite the constants a_n. (You don't need to know the details of this proof.)
  - Be able to find Taylor series and manipulate series. Often you can find Taylor series without computing the coefficients, but instead using other series that you already know: See the homework problems #1, #2, #7, #8 on page 188.
  - Understand that if f is analytic on a circle (centered at z_o), its Taylor series must converge to f on that circle. (This is what is proved in Taylor's theorem!) Conversely, the largest circle (centered at z_o) on which the function is analytic is also the largest circle on which the Taylor series converges to f. See the homework problem #10, page 188.
- *Laurent's Theorem*
  - What are the formulas for the coefficients in a Laurent series?
  - Laurent series are usually found by using other means than computing the coefficients (often using known Taylor series such as $e^w = \sum_{n=0}^{\infty} \frac{w^n}{n!}$ or $\frac{1}{1-w} = \sum_{n=0}^{\infty} w^n (\text{for } |w| < 1)$). See the examples from class (Feb. 7th), the examples in §56, and the problems on page 198 (#2 and #7 are on the homework, but #1–#7 are all similar examples).