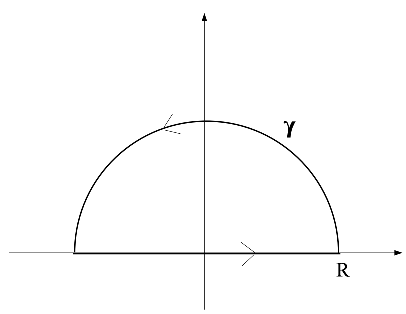
Math 122B: Complex Variables

Types of Real Integrals

I. Integrals of the form P.V. $\int_{-\infty}^{\infty} \frac{\mathbf{p(x)}}{\mathbf{q(x)}}\,\mathbf{dx}$ where $\mathbf{p(x)}$ and $\mathbf{q(x)}$ are polynomials and $\mathbf{q(x)}$ has no zeros (for $-\infty < \mathbf{x} < \infty$)

We will consider the complex function $f(z) = \dfrac{p(z)}{q(z)}$ and evaluate its integral along the following contour γ:



which consists of the line L_R $(-R < x < R)$ and of $\widehat{C}_R$, the upper-half circle centered at the origin.

Let R be large enough that all poles of $f(z)$ in the upper-half plane $(z_1, ..., z_n)$ are located inside the closed contour γ. Then, using residue theory,

$$\int_\gamma f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z).$$

The value of the integral is found by splitting the integral over γ into two parts. In the limit as $R \to \infty$, the integral over L_R will give the desired real integral, and the integral over $\widehat{C}_R$ will go to zero. Following this plan, let α be the constant $2\pi i \sum_{k=1}^{n} \operatorname{Res}_{z=z_k} f(z)$, so the above equality becomes

$$\int_{L_R} f(z)\,dz + \int_{\widehat{C}_R} f(z)\,dz = \alpha.$$

Then, parameterizing L_R by $z = x$ $(-R < x < R)$, we have

$$\int_{-R}^{R} f(x)\,dx + \int_{\widehat{C}_R} f(z)\,dz = \alpha.$$

Finally, we take the limit of both sides as $R \to \infty$:

$$\int_{-\infty}^{\infty} f(x)\,dx + \lim_{R\to\infty} \int_{\widehat{C}_R} f(z)\,dz = \alpha.$$

All that remains is to show that $\int_{\widehat{C}_R} f(z)\,dz \to 0$. This is done by estimating the integral as follows:

$$\left|\int_{\widehat{C}_R} f(z)\,dz\right| \le |\widehat{C}_R| \max_{z\in\widehat{C}_R} |f(z)| \le \pi R \max_{|z|=R} |f(z)|$$

Since $f(z) = \dfrac{p(z)}{q(z)}$, we estimate $|f(z)|$ by using the triangle inequality to bound $|p(z)|$, but in order to bound $\dfrac{1}{|q(z)|}$, remember that we need $|q(z)|$ *larger* than something! As an example, let $p(z) = z^2 + 1$ and let $q(z) = 2z^6 + 3$. Using the triangle inequality, $|p(z)| = |z^2 + 1| \le |z|^2 + 1$, which equals $R^2 + 1$ on the circle $|z| = R$. For the denominator, use the inequality $|2z^6 + 3| \ge |2|z|^6 - 3|$, which equals $2R^6 - 3$ on the circle $|z| = R$ (if $R^6 > \frac{3}{2}$). This implies that $\dfrac{1}{|2z^6+3|} \le \dfrac{1}{2R^6 - 3}$ on the circle. If our final bound for the integral is something that goes to zero as R goes to infinity — that is,

$$\left|\int_{\widehat{C}_R} f(z)\,dz\right| \le F(R) \to 0 \text{ as } R \to \infty$$

— then, the limit of the integral must be 0. (In the example above, $F(R) = \dfrac{R^2+1}{2R^6-3}$.) Finally, using this we have found the value of the real integral:

$$\int_{-\infty}^{\infty} f(x)\,dx = \alpha.$$

II. Integrals of the form P.V.$\int_{-\infty}^{\infty}$ f(x)sin(x) dx (or P.V.$\int_{-\infty}^{\infty}$ f(x)cos(x) dx) where f(x) has no singularities on the real axis.

As before, we will integrate a complex function, but we won't simply replace x with z. Instead, we consider the complex function

$$g(z) = f(z)\,e^{iz} = f(z)\cos(z) + i f(z)\sin(z).$$

Using the same contour γ (with R large enough that all poles of $f(z)$ in the upper-half plane $(z_1, ..., z_k)$ are inside γ),

$$\int_\gamma f(z)e^{iz}\,dz = \alpha \qquad \text{where } \alpha = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} g(z).$$

Breaking up the integral over γ into two parts, we have

$$\int_{-R}^{R} f(x)\cos(x)\,dx + i\int_{-R}^{R} f(x)\sin(x)\,dx + \int_{\widehat{C}_R} f(z)e^{iz}\,dz = \alpha.$$

Again, we only need to take the limit as $R \to \infty$ and show that $\int_{\widehat{C}_R} f(z)e^{iz}\,dz \to 0$. Then, matching up the real and imaginary parts of both sides,

$$\int_{-\infty}^{\infty} f(x)\cos(x)\,dx = \text{Re }\alpha \quad \text{and} \quad \int_{-\infty}^{\infty} f(x)\sin(x)\,dx = \text{Im }\alpha.$$

To show that the integral over the half-circle disappears in the limit, we often need **Jordan's inequality**: $\boxed{\int_0^{2\pi} e^{-R\sin(\theta)}\,d\theta < \frac{\pi}{R}}$ (for $R > 0$). If f is a polynomial with large enough power in the denominator (so with enough decay at infinity), we can use the easier method of bounding the integral, as in II — but if not, we require Jordan's inequality. For example, consider $\int_{\widehat{C}_R} \frac{ze^{iz}}{z^2+1}\,dz$. Parameterizing the upper-half circle, we have

$$\left|\int_{\widehat{C}_R} \frac{ze^{iz}}{z^2+1}\,dz\right| = \left|\int_0^{2\pi} \frac{(Re^{i\theta})e^{iRe^{i\theta}}}{R^2e^{2i\theta}+1}\,iRe^{i\theta}\,d\theta\right| \leq \frac{R^2}{R^2-1}\int_0^{2\pi} e^{-R\sin\theta}d\theta \leq \frac{\pi R}{R^2-1}.$$

In the last step, we used Jordan's inequality... without this extra decay from the integral, we can't prove that the right-hand side goes to zero!

III. Integrals of the form $\int_0^{2\pi} \mathbf{F}(\cos\,\theta, \sin\,\theta)\,\mathbf{d}\theta$, where $\mathbf{F}(\frac{\mathbf{z}+\mathbf{z}^{-1}}{\mathbf{2}}, \frac{\mathbf{z}-\mathbf{z}^{-1}}{\mathbf{2i}})$ has no poles on the unit circle.

Notes: The integral may be taken over any interval of length 2π since the function is periodic. Also, you may see integrals where, for example, θ goes from 0 to π — by symmetry, these often equal one-half the value of the integral from 0 to 2π!

The integral in this case is really a parameterized version of a contour integral on the unit circle. Parameterizing the unit circle by $z(\theta) = e^{i\theta}$ ($0 \leq \theta \leq 2\pi$ — or any other interval of length 2π!), we see that *on the unit circle* $\cos(\theta)$ and $\sin(\theta)$ can be written in terms of z as

$$\cos(\theta) = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{z+z^{-1}}{2} \quad \text{and} \quad \sin(\theta) = \frac{e^{i\theta}-e^{-i\theta}}{2i} = \frac{z-z^{-1}}{2i}.$$

Therefore,

$$\int_0^{2\pi} F(\cos\theta, \sin\theta)\,d\theta = \int_{C_1} F(\tfrac{z+z^{-1}}{2}, \tfrac{z-z^{-1}}{2i})\,dz = 2\pi i\sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} F(\tfrac{z+z^{-1}}{2}, \tfrac{z-z^{-1}}{2i}).$$

Make sure the residues in the sum are only for the poles *inside* the unit circle!!

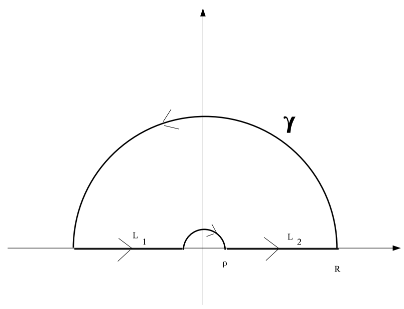
IV. Integrals that require different contours.

Notes: These are integrals for which the complex function we want to integrate has a pole or a branch point somewhere on the real axis. In either cases, a contour that avoids going through the pole or the branch cut is needed!

Examples:

- $\displaystyle\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$

This is similar to case II, except that the complex function $g(z) = \dfrac{e^{iz}}{z}$ has a pole at $z = 0$! Consider the following contour $\gamma = L_1 \cup (-\widehat{C}_\rho) \cup L_2 \cup \widehat{C}_R$ (where $R > \rho > 0$):



The function is analytic inside the contour, so for any $R > \rho > 0$,

$$\int_\gamma \frac{e^{iz}}{z}\,dz = 0.$$

Splitting the integral up, and parameterizing $-L_1$ by $z = -r\,(\rho < r < R)$ and L_2 by $z = r\,(\rho < r < R)$,

$$-\int_\rho^R \frac{e^{-ir}}{-r}(-dr) - \int_{\widehat{C}_\rho} \frac{e^{iz}}{z}\,dz + \int_\rho^R \frac{e^{ir}}{r}\,dr + \int_{\widehat{C}_R} \frac{e^{iz}}{z}\,dz = 0.$$

Using Jordan's inequality, we can show that $\lim\limits_{R\to\infty} \int_{\widehat{C}_R} \frac{e^{iz}}{z}\,dz = 0$. Since $z = 0$ is a simple pole of g, we will use the Laurent series of g around 0 to show that

$$\lim_{\rho\to 0} \int_{\widehat{C}_\rho} \frac{e^{iz}}{z}\,dz = i\pi \operatorname*{Res}_{z=0} g(z).$$

The Laurent series of $g(z)$ is $\dfrac{b_o}{z} + \displaystyle\sum_{n=0}^\infty a_n z^n$, where $b_o = \operatorname*{Res}\limits_{z=0} g(z)$. Then,

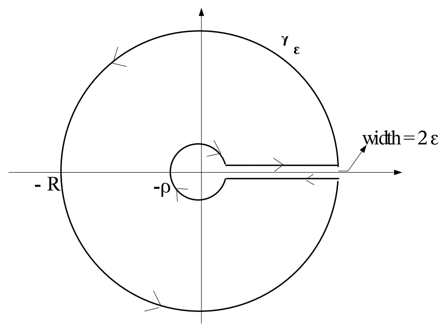
$$\begin{aligned}\int_{\widehat{C}_\rho} \frac{e^{iz}}{z}\,dz &= \int_{\widehat{C}_\rho} \frac{b_o}{z}\,dz + \sum_{n=0}^\infty a_n \int_{\widehat{C}_\rho} z^n\,dz = \int_0^\pi \frac{i\rho e^{i\theta}}{\rho e^{i\theta}}\,d\theta + \sum_{n=0}^\infty a_n \int_{\widehat{C}_\rho} z^n\,dz \\ &= i\pi + \sum_{n=0}^\infty a_n \int_{\widehat{C}_\rho} z^n\,dz.\end{aligned}$$

Simply show that the last sum tends to zero (notice that each of the integrals in the sum can be bounded by $\pi\rho^{n+1}$!). Using these limits (notice that $b_o = 1$ in this case; also $e^{ir} - e^{-ir} = 2i\sin(r)$),

$$2i\int_0^\infty \frac{\sin r}{r}\,dr = i\pi.$$

- $\displaystyle \int_0^\infty \frac{\mathbf{1}}{\sqrt{\mathbf{x}}(\mathbf{x}+\mathbf{1})}\,\mathbf{dx} = \pi$

Notice that $\sqrt{x}$ is simply the positive square root of a postive real number, but when we consider the complex function $f(z) = \dfrac{z^{-\frac{1}{2}}}{(z+1)}$ specifiying the branch cut is necessary! We shall use the branch $0 < \arg z < 2\pi$; therefore, $f(z)$ is analytic away from the branch cut $\arg z = 0$ and away from the pole at $z = -1$. We will use the contour γ_ϵ:



We will take the limit $\epsilon \to 0$ before taking limits in ρ and R; therefore, it is fine to simplify by considering the contour as pictured on page 274 of your book where $\epsilon = 0$. The interesting part is what happens on the two lines: L_1 is parameterized by $z(x) = x + i\epsilon$, and $-L_2$ is parameterized by $z(x) = x - i\epsilon$ (with $\sqrt{\rho^2-\epsilon^2} \le x \le \sqrt{R^2-\epsilon^2}$ — of course, this will just become $\rho \le x \le R$ when we take the limit $\epsilon \to 0$.)

$$\int_{L_1} f(z)\,dz = \int_{\sqrt{\rho^2-\epsilon^2}}^{\sqrt{R^2-\epsilon^2}} \frac{(x+i\epsilon)^{-\frac{1}{2}}}{(x+i\epsilon)+1}\,dx \quad \text{and} \quad \int_{L_2} f(z)\,dz = \int_{\sqrt{\rho^2-\epsilon^2}}^{\sqrt{R^2-\epsilon^2}} \frac{(x-i\epsilon)^{-\frac{1}{2}}}{(x-i\epsilon)+1}\,dx$$

Make sure to use the branch cut when evaluating $(x \pm i\epsilon)^{-\frac{1}{2}}$! Since $x + i\epsilon$ is in the first quadrant, but $x - i\epsilon$ is in the fourth, when we write these in polar coordinates using their arguments, we have

$$(x+i\epsilon) = \sqrt{x^2+\epsilon^2}\, e^{i\arg(x+i\epsilon)} \longrightarrow x \cdot e^0 \text{ as } \epsilon \to 0$$
$$(x-i\epsilon) = \sqrt{x^2+\epsilon^2}\, e^{i\arg(x-i\epsilon)} \longrightarrow x \cdot e^{2\pi i} \text{ as } \epsilon \to 0.$$

Therefore, as $\epsilon \to 0$,

$$\int_{L_1} f(z)\,dz \to \int_\rho^R \frac{\frac{1}{\sqrt{x}}}{(x+1)}\,dx$$
$$\int_{L_2} f(z)\,dz \to -\int_\rho^R \frac{\frac{1}{\sqrt{x}}e^{-i\pi}}{(x+1)}\,dx.$$

In the book, these are given as equalities since they assume the limit has already been taken – that is, L_1 is the line on the real axis, but with all limits being taken from above the line, and L_2 is the line on the real axis with the limits taken from below. Finally, let $\rho \to 0$ and $R \to \infty$ (and estimate to show that the integrals of f on both $\widehat{C}_\rho$ and $\widehat{C}_R$ go to zero!)

$$2\int_0^\infty \frac{1}{\sqrt{x}(x+1)}\,dx = 2\pi i \operatorname*{Res}_{z=-1} f(z) = 2\pi i\left((-1)^{-\frac{1}{2}}\right) = 2\pi i\left(e^{-i\frac{\pi}{2}}\right) = 2\pi$$