Math 122B: Complex Variables

Review

The field of complex numbers is defined as the set of ordered pairs of real numbers with the addition and multiplication operations defined by

$$(x, y) + (v, w) = (x + v, y + w)$$
$$(x, y) \cdot (v, w) = (xv - yw, xw + yv).$$

The zero element is $(0, 0)$ and the identity element is $(1, 0)$. If we identify every real number x with the complex number $(x, 0)$, then in terms of the special number $i := (0, 1)$, we can represent any complex number $z = (x, y)$ as $z = x + iy$.

A function $f(z)$ of a complex variable z is said to be *differentiable* at a point z_o with derivative

$$f'(z_o) = \lim_{\Delta z \to 0} \frac{f(z_o + \Delta z) - f(z_o)}{\Delta z}$$

provided the above limit exists.

As a consequence of this definition, whenever a function $f(z) = u(z) + iv(z)$ is differentiable at a point $z_o = x_o + iy_o$, the *Cauchy-Riemann equations* for the real-valued functions $u(z)$ and $v(z)$ must hold at z_o.

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}; \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}$$

We will interchangably use the symbol u for the function $u(z)$ of a complex variable z or the function $u(x, y)$ of two real variables x and y (where of course $u(x, y) = u(x + iy)$). If f is differentiable everywhere, f is called *analytic* in the complex plane or *entire*. In this case, $u_x = v_y$ and $u_y = -v_x$ at all points; notice this implies that both u and v are harmonic! Moreover, the derivative of f at a point $z = x + iy$ can be computed in terms of the partial derivatives of u and v with respect to x (or y):

$$f'(z) = u_x(x, y) + iv_x(x, y) \, (= v_y(x, y) - iu_y(x, y)).$$

There are functions $u(x, y)$ and $v(x, y)$ such that the Cauchy-Riemann equations hold but $u(z) + iv(z)$ is *not* differentiable! To guarantee that $u(z) + iv(z)$ is differentiable, we need the first-order partial derivatives u_x, u_y, v_x, and v_y to exist and to satisfy the Cauchy-Riemann equations and also to be continuous.

If a function $f(z)$ is analytic on a set S in the complex plane (i.e., $f(z)$ is differentiable at each point in some open set containing S), one consequence is the Cauchy-Riemann equations.

Another interesting consequence that we will prove this quarter is that if there are enough points in the set S where $f(z) = 0$, then $f(z) \equiv 0$ everywhere on S. "Enough points" basically means that the points pile up somewhere in S. Certainly, if $f(z) = 0$ for all points on some line segment in S, then $f \equiv 0$ in S. For another example, if f is an entire function such that $f(\frac{1}{n}) = 0$ for all positive integers n, then the function must be 0 everywhere in the complex plane! (The points $1/n$ pile up at 0.) However, it is possible to have $f(n) = 0$ for all integers n and still have $f(z) \not\equiv 0$.

The exponential function is the entire function defined by

$$\boxed{e^z := e^x e^{iy} = e^x \cos(y) + ie^x \sin(y)}$$

for all complex numbers $z = x + iy$. (Alternatively, e^z could have been defined by the familiar infinite power series $\sum_{j=0}^{\infty} \frac{z^j}{j!}$ – we will prove later that for every z, this sum yields the complex number e^z defined above.)

With this definition, we can compute that $e^{z_1} e^{z_2} = e^{z_1+z_2}$ and therefore that

$$(\cos(z) + i\sin(z))^n = \cos(nz) + i\sin(nz).$$

Also, it is clear that e^z is entire, is never equal to 0, and is periodic in y with period 2π.

We would like to define the logarithm as the inverse of the exponential function. Therefore, for each fixed z, we want to solve $e^w = z$ to find $w = \log(z)$. Using polar coordinates, let $z = re^{i\theta}$ for some $r > 0$ and θ. In order to solve $e^w = e^{u+iv} = e^u e^{iv} = re^{i\theta}$, we clearly need $u = \ln(r)$. However, since e^{iv} is periodic in v, there will be many values of v (differing by multiples of 2π) such that $e^{iv} = e^{i\theta}$. Therefore, we define $\log(z)$ as the multi-valued function $\log(z) = \ln(r) + i\arg(z)$.

If we want unique values of the logarithm, we must make the choice of which v to use. A *branch* of $\log(z)$ is a restriction on the angle $\theta = \arg(z)$ that makes $\log(z)$ single-valued. For example, the *principle branch* of $\log(z)$ is denoted $\mathrm{Log}(z)$ and is defined such that $-\pi < \theta < \pi$.

$$\boxed{\mathrm{Log}(z) = \ln r + i\theta \text{ for } -\pi < \theta < \pi}$$

We can prove that any branch of $\log(z)$ (i.e., $\log(z) = \ln r + i\theta$ where $\alpha < \theta < 2\pi + \alpha$ for some α) is analytic (away from its *branch cut* – the line from the origin with angle α) and that the derivative is given by

$$\frac{\mathrm{d}}{\mathrm{d}z} \log(z) = \frac{1}{z}.$$

Powers of z are defined using the exponential and logarithm functions.

$$\boxed{z^w := \exp(w \log(z))}$$

This definition coincides with the usual definition of x^y where x and y are real numbers. Typically, z^w will be multi-valued. For example, we can compute that $i^i = e^{-arg(i)} = e^{-(4n+1)*\pi/2}$ for $n = 0, \pm 1, \pm 2, ...$

The trigonometric functions are entire functions defined in terms of the exponential function:

$$\sin(z) := \frac{e^{iz} - e^{-iz}}{2i}; \qquad \cos(z) := \frac{e^{iz} + e^{-iz}}{2}$$

As with e^z, the inverses of $\sin(z)$ and $\cos(z)$ are multi-valued since these functions are periodic (in x!)