Math 122B: Complex Variables

Singularities

If z_o is an isolated singularity of the function $f(z)$, we can draw some small circle around z_o with radius $\epsilon > 0$ such that $f(z)$ is analytic in $0 < |z - z_o| < \epsilon$. By Laurent's theorem, $f(z)$ has a Laurent expansion in this domain. There are three cases: either the Laurent series has no singular terms, finitely many singular terms, or infinitely many singular terms. These three cases define whether z_o is a *removable singularity*, *pole* (with order equal to the power of the first singular term in the series), or *essential singularity*.

Fact. Let z_o be an isolated singularity of f. Then z_o is a pole of order m if and only if $f(z) = \dfrac{\phi(z)}{(z - z_o)^m}$ for some function ϕ that is analytic at z_o.

Note. Considering the Taylor series of $\phi(z)$ makes it easy to compute residues at poles!

Example. The function $f(z) = \dfrac{e^z}{(z^2 + 9)^5(z + 7)}$ has an isolated singularity at $z_o = 3i$. We could compute the entire Laurent series around $3i$, but it would be a lot of work. If we just want the residue at $3i$, notice that $\phi(z) = \dfrac{e^z}{(z + 3i)^5(z + 7)}$ is analytic at $3i$, so it must have a Taylor series:

$$\begin{aligned} f(z) = \frac{\phi(z)}{(z - 3i)^5} &= \frac{1}{(z - 3i)^5} \left(\sum_{n=0}^{\infty} a_n (z - 3i)^n \right) \\ &= \sum_{n=5}^{\infty} a_n (z - 3i)^{n-5} + \frac{a_4}{z - 3i} + ... + \frac{a_1}{(z - 3i)^4} + \frac{a_o}{(z - 3i)^5} \end{aligned}$$

The residue of $f(z)$ at $3i$ is therefore the constant

$$\operatorname*{Res}_{z=3i} f(z) = a_4 = \frac{\phi''''(3i)}{4!}.$$

All we need to do is find $\phi''''(3i)$....

We also discussed zeros of an analytic function, and the relationship between zeros of $f(z)$ and poles of $\dfrac{1}{f(z)}$. This can be useful in computing residues of $\dfrac{p(z)}{q(z)}$ where p and q are polynomials.

Definition. An analytic function $f(z)$ has a *zero* of order m at z_o if

$$f(z_o) = 0, \;\; f'(z_o) = 0, \; ... \;, \;\; f^{(m-1)}(z_o) = 0, \text{ and } \; f^{(m)}(z_o) \neq 0.$$

Notes on zeros and poles.

• We proved in class that $f(z)$ has a zero of order m at z_o if and only if $f(z) = (z - z_o)^m g(z)$ for some function g that is analytic at z_o with $g(z_o) \neq 0$.

• We also showed $f(z)$ has a zero of order m at z_o if and only if $\dfrac{1}{f(z)}$ has a pole of order m at z_o.

• Consider $\dfrac{p(z)}{q(z)}$ where $p(z_o) \neq 0$ and z_o is a zero of order m of the function q: Then, z_o is a pole of order m of the function $\dfrac{p(z)}{q(z)}$. To compute the residue at z_o, write

$$\frac{p(z)}{q(z)} = \frac{p(z)/g(z)}{(z - z_o)^m}$$

where $p(z)/g(z)$ is analytic at z_o. The residue will be a_{m-1}, the $(m-1)$th term in the Taylor expansion of $p(z)/g(z)$. In particular, if $m = 1$,

$$\operatorname*{Res}_{z=z_o} \frac{p(z)}{q(z)} = \frac{p(z_o)}{g(z_o)} = \frac{p(z_o)}{q'(z_o)}.$$

The last equality follows since $q(z) = (z - z_o)g(z)$ implies that $q'(z) = g(z) + (z - z_o)g'(z)$. Therefore, $q'(z_o) = g(z_o)$. This can be a very quick way to compute residues, but notice that it is just a special case of the method for computing residues at poles – take $\phi(z) = p(z)/g(z)$.

• We also proved that fact that an analytic function can only have isolated zeros. One corollary of this is that in a closed and bounded domain, f has only finitely many zeros.

We also have the following theorems regarding removable and essential singularities.

Theorem. Let f be analytic and bounded in $0 < |z - z_o| < \epsilon$. Then either f is analytic at z_o or z_o is a removable singularity of f.

Proof. We know that ____,

and that ____.

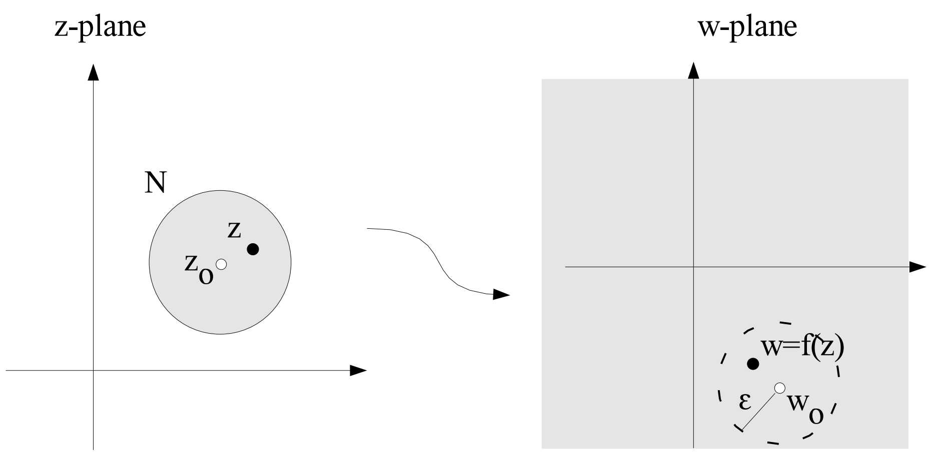
We want to show that ____.

We mentioned that a function with an essential singularity at z_o must take on every complex values (excepting possibly one) infinitely many times in every neighborhood of z_o. (This is *Picard's theorem* – in other words, consider the mapping $w = f(z)$: If we start with any neighborhood N of z_o, no matter how small, N will map to the entire $w-$plane, excepting possibly one point). We will prove Weierstrauss' theorem, which is a weaker version of this. The difference is that the theorem below only states that the function must get *close* to every possible complex value.

Weierstrauss' Theorem. Suppose that z_o is an essential singularity of a function f and let w_o be any complex number. Pick any deleted neighborhood N of z_o. Then, for any $\epsilon > 0$,

$$|f(z) - w_o| < \epsilon \text{ for some } z \in N.$$

Picture. Pick any w_o, any N, and any ϵ. There must exist $z \in N$ as shown.



Proof. (*by contradiction*)
Assume that there is a number $\epsilon > 0$ such that ______________________________
for all __________. Let

$$g(z) = \frac{1}{f(z) - w_o}.$$

The function g is defined and analytic on the deleted neighborhood N. We know that $g(z)$ is

________________________; therefore, z_o is a ________________________ of g.

Replace g with the analytic function $\tilde{g}$ [$\tilde{g}(z) = g(z)$ on N]. Either $\tilde{g}(z_o) \neq 0$ or z_o is a zero of order m of $\tilde{g}$. Since $f(z) = \frac{1}{g(z)} + w_o$ on N, these two cases imply that either z_o is a removable singularity of f or z_o is a pole of order m of f. Both contradict the assumption that z_o is an essential singularity of f.