Math 122B: Complex Variables

Uniqueness of Taylor Series

The major goal of sections 57 through 60 in the book is to prove that the Taylor series representation of an analytic function is unique. The main result is that if you have a **power series** of the form

$$\sum_{n=0}^{\infty} a_n (z - z_o)^n$$

that converges to a function $f(z)$, then the function is analytic and the power series must actually be its Taylor series about the point z_o! In particular, the coefficients a_n must be equal to $\frac{f^{(n)}(z_o)}{n!}$ for each $n = 0, 1, 2, ...$

If the power series converges for *all* z, then the function $f(z)$ is entire. If not there must be a largest circle on which the power series converges. If R is the largest radius such that the series converges for all $|z - z_o| < R$, then R is called the **radius of convergence** (and $C_R(z_o)$ is called the *circle of convergence*). The function f is analytic for $|z - z_o| < R$, but must fail to be analytic at at least one point on $C_R(z_o)$.

§57 Uniform Convergence of Power Series

A power series $\sum_{n=0}^{\infty} a_n(z - z_o)^n$ converges **absolutely** at a point z if $\sum_{n=0}^{\infty} |a_n(z - z_o)^n|$ converges.

Recall that the power series converges (pointwise!) in a closed set if

$$\text{For } \textit{every } z \text{ in the set,} \quad \left| \sum_{n=N+1}^{\infty} a_n(z - z_o)^n \right| \to 0 \text{ as } N \to \infty.$$

It converges **uniformly** on a closed set if

$$\max_{z \text{ in the set}} \left| \sum_{n=N+1}^{\infty} a_n(z - z_o)^n \right| \to 0.$$

The main theorems in this section state that if a power series converges, it converges both absolutely and uniformly:

Theorem. If the infinite series $\sum_{n=0}^{\infty} a_n(z - z_o)^n$ has circle of convergence $|z - z_o| = R$, then for any $0 < r < R$, the series is uniformly convergent in the closed disk $|z - z_o| \le r$. (It is also absolutely convergent for each point in $|z - z_o| \le r$.

Notice that this theorem only true for *complex* power series! If we had a real power series that converges to a real function $g(x)$, it is not necessarily uniformly convergent. Example: .... Therefore, the subsequent results in this chapter that follow from this theorem are also only true for complex power series.

§58 Continuity of Power Series

It is a well-known fact from analysis that uniform convergence of a power series implies that the sum is a continuous function! (This is very general; in particular, it's true for both real and complex series.) Since we know that complex power series always converge uniformly inside the circle of convergence we have the following theorem:

Theorem: Power series are Continuous. If $S(z) = \displaystyle\sum_{n=0}^{\infty} a_n(z - z_o)^n$ is a power series with radius of convergence R, then $S(z)$ is continuous for all $|z - z_o| < R$.

Notes. The proof is an "$\epsilon/3$" argument: we are required to show that $|S(z) - S(w)| < \epsilon$ for all w close enough to w, so we use the triangle inequality

$$|S(z) - S(w)| \leq |S(z) - S_N(z)| + |S_N(z) - S_N(w)| + |S_N(w) - S(w)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

To prove the first and last terms are less than $\epsilon/3$ requires uniform convergence!

§59 Integration and Differentiation of Power Series

Another great thing about uniform convergence is that we are allowed to differentiate the series term by term! ?? For a complex series, it's easier to prove this by first showing a complex power series can be integrated term by term, and is therefore analytic! ?? We must first show that the power series has a derivative (i.e., is analytic)!

Theorem: Integration of Power Series. Let C be any contour inside the circle of convergence of a power series $S(z) = \sum_{n=0}^{\infty} a_n(z - z_o)^n$. Let $g(z)$ be any function that is continuous on C. Then,

$$\int_C g(z)S(z)\,dz = \sum_{n=0}^{\infty} a_n \int_C g(z)(z - z_o)^n\,dz$$

Notes. The integral above exists since $S(z)$ is continuous. To prove the equality, you can simply look at the partial sums and take a limit. Also, in the case $g(z) \equiv 1$, this theorem really just says that to integrate a power series $S(z)$, you can integrate term by term!

Corollary: Power Series are Analytic. The sum $S(z)$ is analytic at each point z inside the circle of convergence $B_R(z_o)$.

Proof. Take $g(z) \equiv 1$. Since the powers of $(z - z_o)$ are all analytic functions, the above theorem shows that for any *closed* contour C inside the circle of convergence,

$$\int_C S(z)\,dz = 0.$$

This implies that $S(z)$ is an analytic function in the domain $|z_o - z| < R$. (This is the converse of Cauchy-Goursat – see §48.) □

Since $S(z)$ is analytic, we can take its derivative. It turns out that we can also differentiate $S(z)$ term by term!

Theorem: Differentiation of Power Series. The power series $S(z)$ can be differentiated term by term: For each z inside the circle of convergence

$$S'(z) = \sum_{n=1}^{\infty} n a_n (z - z_o)^{n-1}.$$

If we know a power series representation for a function, this theorem allows us to find power series representations for all of the function's derivatives!

Example. Since we know that

$$\frac{1}{z+2} = \frac{1}{1-[-(z+1)]} \sum_{n=0}^{\infty} (-1)^n (z+1)^n$$

is a power series representation for $\frac{1}{z+2}$ around $z_o = -1$ with radius of convergence $R = 1$ (notice the series converges inside the circle $C_1(-1)$!), then

$$-\frac{1}{(z+2)^2} = \sum_{n=1}^{\infty} (-1)^n n (z+1)^{n-1} = \sum_{n=0}^{\infty} (-1)^{n+1} (n+1)(z+1)^n \tag{1}$$

is a power series representation about -1 for the function $h(z) = -\frac{1}{(z+2)^2}$ (Question: Is this really the Taylor series? Compute $\frac{h^{(n)}(-1)}{n!}$ to find out!) You can also find a power series for $h'(z) = \frac{1}{(z+2)^3}$ and for further derivatives of $h(z)$.

§60 Uniqueness of Taylor Series

We now show that if we have a power series representation for a function, it *must* be the Taylor series representation. In other words, in the previous example, once we know that equation (1) is true, we know that the power series *is* the Taylor series for h about the point -1 without needing to check whether the series has the "right" coefficients!

Theorem: Uniqueness of Taylor Series. If for some $R > 0$ a power series

$$\sum_{n=0}^{\infty} a_n (z - z_o)^n$$

converges to $f(z)$ for all $|z - z_o| < R$, then this series is the Taylor series expansion for f about the point z_o.

Proof. Since we know that (for $|z - z_o| < R$)

$$f(z) = \sum_{m=0}^{\infty} a_m (z - z_o)^m,$$

we can use the theorem on integration of power series: For any radius $0 < r < R$ and for any function g that is continuous on $C_r(z_o)$,

$$\int_{C_r(z_o)} g(z) f(z)\, dz = \sum_{m=0}^{\infty} a_m \int_{C_r(z_o)} g(z)(z - z_o)^m\, dz. \tag{2}$$

For each $n = 0, 1, 2, ...$, we can let $g(z) = \dfrac{1}{2\pi i} \dfrac{1}{(z - z_o)^{n+1}}$. An elementary computation shows that

$$\frac{1}{2\pi i} \int_{C_r(z_o)} \frac{(z - z_o)^m}{(z - z_o)^{n+1}} \, dz = \frac{1}{2\pi i} \int_{C_r(z_o)} \frac{1}{(z - z_o)^{n-m+1}} \, dz = \begin{cases} 1 & \text{for } m = n \\ 0 & \text{for } m \neq n \end{cases}$$

(See §40, Exercise #10.) Using this and Cauchy's integral formula, (2) becomes

$$\frac{f^{(n)}(z_o)}{n!} = \sum_{m=0}^{\infty} a_m \frac{1}{2\pi i} \int_{C_r(z_o)} \frac{1}{(z - z_o)^{n-m+1}} \, dz = a_n.$$

Therefore, the given power series is exactly equal to the Taylor series for f about the point z_o. $\square$

Similarly, the Laurant series expansion is unique. (See §60, Theorem 2.)