Final Review

Math 124A: PDEs

First-order and Diffusion Equations

Review your old homeworks, and look over the midterm review and the midterm. In general, also make sure you're comfortable with all the calculus techniques we've been using this quarter, including the delta function, integration by parts, change of variables in integrals, and the chain rule.

§2.1, 2.2 The Wave Equation

- Characteristic coordinates
  - What are the characteristic coordinates ξ and η for the wave equation? How do you show that in these coordinates, the wave equation becomes $u_{\xi\eta} = 0$? Also remember how to integrate this equation (don't forget the arbitrary functions of one variable!).
- D'Alembert's formula
  - Example: Solve $u_{tt} = u_{xx}$ with the initial conditions $u(x, 0) = \cos(x)$ and $u_t(x, 0) = xe^{-x^2}$.
- Causality
  - Domain of dependence (domain of influence)
  - Characteristic lines for the wave equation ($x \pm ct = \textit{constant}$)
  - Example: If the initial conditions $u(x, 0)$ and $u_t(x, 0)$ are both 0 everywhere outside of $[-M, N]$ for some positive numbers M and N, in what region of the domain (for $t > 0$) must the solution of the wave equation be 0? (Give inequalities that describe the region).
- Energy method
  - Know how to derive the fact that the energy for the wave equation is a constant.
  - Example: Prove uniqueness of the initial value problem for the wave equation on the whole line using the law of conservation of energy. (That is, assume u_1 and u_2 solve the wave equation with the same initial conditions. Prove $u_1 = u_2$ using the energy.) Hint: You may assume that if f is positive and $\int_a^b f(x) = 0$, then $f(x) = 0$ on the interval $a < x < b$.

§3.2 Reflections of Waves

- Wave on the half-line $0 < x < \infty$
  - For Dirichlet boundary condition $u(0,t) = 0$, understand why taking the odd extension of the initial conditions and using d'Alembert's formula gives the right solution on the domain $0 < x < \infty$.
  - For the Neumann boundary condition $u_x(0,t) = 0$, use the even extension of the initial conditions.
  - Examples: See the last homework.
- Wave equation on the finite interval $[0, l]$
  - Solve this by extending the initial conditions and using d'Alembert's formula. Notice the formula for the solution will be different in each region in Figure 6 in §3.2.
  - What is the right extension if $u(0,t) = u(l,t) = 0$? What if $u_x(0,t) = 0$ and $u(l,t) = 0$ (see Exercise #10)?
  - Examples: Exercise #10 and #11 in §3.2

§3.4 Waves with a Source

- Know the formula to solve the wave equation with a source

$$\begin{aligned} &u_t - c^2 u_{xx} = f(x,t) && \text{on } -\infty < x < \infty \\ &u(x,0) = \phi(x);\ u_t(x,0) = \psi(x). \end{aligned}$$

  Example: Solve the problem above if $f(x,t) = xe^{-x}$, $\phi(x) = e^{-x^2}$ and $\psi(x) = \sin(x)$.

- Example: Solve the equation $u_{tt} - 2x = 4u_{xx}$ on the half-line with the Dirichlet boundary condition $u(0,t) = e^{-t}$ and the initial conditions $u(x,0) = 1$; $u_t(x,0) = \cos(x)$. Hint: Let $v(x,t) = u(x,t) - e^{-t}$. What problem does v solve? You will need to use different formulas to compute v in the regions $x > ct$ and $x < ct$.
- Example: Exercise #13 in §3.4