Math 124a: Final

Name Solutions

Formulas

$$S(x,t) = \frac{1}{\sqrt{4\pi kt}} e^{-\frac{x^2}{4kt}} \qquad \mathscr{E}\mathrm{rf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-p^2}\, dp$$

For any constant a, the solution of

$$\begin{cases} u_t = k u_{xx} & \text{for } -\infty < x < \infty \ (t > 0) \\ u(x,0) = ax^2 \end{cases}$$

is $u(x,t) = ax^2 + 2akt$.

For any constant a, the solution of

$$\begin{cases} u_t = k u_{xx} & \text{for } -\infty < x < \infty \ (t > 0) \\ u(x,0) = e^{ax} \end{cases}$$

is $u(x,t) = e^{ax + a^2 kt}$.

For the wave equation on an interval $[a,b]$, the energy at time t is

$$E(t) = \int_a^b \left([u_t(x,t)]^2 + c^2 [u_x(x,t)]^2 \right)\, dx$$

1. [15 pts] Solve $u_x - 2u_y - 5u = 2x^2 - 3xy - 2y^2$ with the condition $u(x,0) = e^x$.

Let $x' = x - 2y$

$y' = 2x + y$

Then $u_x - 2u_y = (u_{x'} + 2u_{y'}) - 2(-2u_{x'} + u_{y'})$

$= 5u_{x'}$

Also $2x^2 - 3xy - 2y^2 = (2x + y)(x - 2y) = y' \cdot x'$

Solve: $5u_{x'} - 5u = x'y'$

Integrating factor: $\underline{e^{-x'}}$ $\Rightarrow 5(e^{-x'}u_{x'} - e^{-x'}u) = e^{-x'}x'y'$

$\Rightarrow 5 \partial/\partial x' (e^{-x'}u) = e^{-x'}x'y'$

$\Rightarrow 5e^{-x'}u = y' \int x'e^{-x'}$

$= y'(-x'e^{-x'} - e^{-x'}) + f(y')$

$\left(\underline{\text{Check integral:}} \ \frac{\partial}{\partial x'}(-x'e^{-x'} - e^{-x'}) = +x'e^{-x'} - e^{-x'} + e^{-x'} \right.$
$\left. = xe^{-x'} \checkmark \right)$

Therefore,

$u = \frac{1}{5}y'(-x' - 1) + \frac{1}{5}e^{-x'}f(y')$

In terms of x and y:

$u(x,y) = -\frac{1}{5}(2x+y)(x-2y+1) + \frac{1}{5}e^{x-2y}f(2x+y)$

Since $u(x,0) = e^{x}$

$$-\frac{1}{5}(2x)(1+x) + \frac{1}{5}e^{x} f(2x) = e^{x}$$

$$\frac{1}{5}e^{x} f(2x) = e^{x} + \frac{1}{5}(2x + 2x^{2})$$

$$f(2x) = 5 + e^{-x}(2x + 2x^{2})$$

$$f(z) = 5 + e^{-z/2}\left(z + z^{2}/2\right)$$

$$\boxed{\begin{aligned} u(x,y) = &-\tfrac{1}{5}\left(2x^{2} - 2y^{2} - 3xy + 2x + y\right) \\ &+ e^{x-2y} + \tfrac{1}{5}\, e^{x-2y}\, e^{(2x+y)/2}\left[2x + y + \frac{(2x+y)^{2}}{2}\right] \end{aligned}}$$

2. [10 pts] Solve for u (in terms of the error function).

$K = 1/4$

$$\overset{+}{=} 4te^{-2x} = f(x,t)$$

$$\begin{cases} u_t - \frac{1}{4}u_{xx} \cancel{-4te^{-2x} = 0} & \text{for } -\infty < x < \infty \\ \phi(x) = u(x,0) = \begin{cases} x & x \geq 0 \\ 0 & x < 0 \end{cases} \end{cases}$$

(DE) with a source

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)\,dy + \int_0^t \int_{-\infty}^{\infty} S(x-y,t-s)[4se^{-2y}]\,dy\,ds$$

$$= \int_{-\infty}^{0} \cancel{\phi}\,dy + \int_0^{\infty} S(x-y,t)\,y\,dy + \int_0^t 4s\Big(\underbrace{\int_{-\infty}^{\infty} S(x-y,t-s)e^{-2y}dy}_{\text{Sol'n of (DE) at time } t-s \text{ with IC } e^{-2x}}\Big)ds$$

$$= \int_0^{\infty} \frac{e^{-(x-y)^2/t}}{\sqrt{\pi t}}\,y\,dy + \int_0^t 4s\,e^{-2x+(t-s)}\,ds$$

COV

$p = \frac{x-y}{\sqrt{t}}$

$dp = -dy/\sqrt{t}$

$y = x - \sqrt{t}\,p$

$$= \frac{1}{\sqrt{\pi}}\int_{x/\sqrt{t}}^{-\infty} e^{-p^2}(x-\sqrt{t}\,p)(-dp) + 4e^{-2x+t}\int_0^t \underbrace{s}_{p}\,\underbrace{e^{-s}ds}_{dq}$$

$$= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{x/\sqrt{t}} e^{-p^2}x\,dp - \frac{\sqrt{t}}{\sqrt{\pi}}\int_{-\infty}^{x/\sqrt{t}} pe^{-p^2}dp + 4e^{-2x+t}\Big[\int_0^t \underbrace{e^{-s}ds}_{-q\ dp} - se^{-s}\Big|_0^t\Big]$$

$$= \frac{x}{2}\Big[\underbrace{\frac{2}{\sqrt{\pi}}\int_{-\infty}^{0} e^{-p^2}dp}_{=1} + \frac{2}{\sqrt{\pi}}\int_0^{x/\sqrt{t}} e^{-p^2}dp\Big] - \frac{\sqrt{t}}{\sqrt{\pi}}\Big(-\frac{e^{-p^2}}{2}\Big|_{-\infty}^{x/\sqrt{t}}\Big)$$

$$+ 4e^{-2x+t}\Big[-e^{-s}\Big|_0^t - te^{-t}\Big]$$

$$u(x,t) = \frac{x}{2} + \frac{x}{2}\,\mathrm{Erf}\Big(\frac{x}{\sqrt{t}}\Big) + \frac{\sqrt{t}}{2\sqrt{\pi}}e^{-x^2/t} + 4e^{-2x}\big[-1+e^{t}-t\big]$$

3. [5+5 pts] Consider the following problem for u:

$$\begin{cases} u_t - ku_{xx} + xe^{-t} = 0 & \text{for } 0 < x < \infty \\ u(x,0) = \delta(x-2) + x^2 + x. \\ u_x(0,t) = e^{-t}. \end{cases}$$

Let $v(x,t) = u(x,t) - xe^{-t}$.
(a) What problem does v solve?
(b) [+5pts] Find the solution u.

(a) Compute: $V_t = u_t + xe^{-t}$

$V_x = u_x - e^{-t}$ $\qquad V_{xx} = u_{xx}$

V solves $\begin{cases} V_t - KV_{xx} = u_t + xe^{-t} - u_{xx} = 0 & \underline{0<x<\infty} \\ V(x,0) = u(x,0) - xe^0 = \delta(x-2) + x^2 \\ V_x(0,t) = u_x(0,t) - e^{-t} = 0 \end{cases}$

(b) V solves (DE) on the half-line with the Neumann BC $V_x(0,t) = 0$, so we can use an even extension of the IC $\phi(x) = \delta(x-2) + x^2$ to solve for V

$$\phi_{Even}(x) = \begin{cases} \delta(x-2) + x^2 & x > 0 \\ \delta(-x-2) + x^2 & x < 0. \end{cases}$$

Therefore,

$$V(x,t) = \int_{-\infty}^{0} S(x-y,t)[\delta(-y-2)+y^2]dy + \int_0^{\infty} S(x-y,t)[\delta(y-2)+y^2]dy$$

$$= S(x+2,t) + S(x-2,t) + \int_{-\infty}^{\infty} S(x-y,t)y^2 dy$$

$$= S(x+2,t) + S(x-2,t) + x^2 + 2Kt$$

$$\Rightarrow u(x,t) = \frac{1}{\sqrt{4\pi Kt}}\left[e^{-(x+2)^2/4Kt} + e^{-(x-2)^2/4Kt}\right] + x^2 + 2Kt + xe^{-t}$$

4. [10 pts] Consider the heat equation on the whole line with an even initial condition:

$$u_t = ku_{xx} \qquad -\infty < x < \infty$$
$$u(x,0) = \phi(x),$$

where ϕ is an even function.
(a) Prove that the solution $u(x,t)$ is an even function of x.
(b) Prove that $u_x(0,t) = 0$.

(a) $u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\,\phi(y)\,dy$

We want to show that $u(-x,t) = u(x,t)$

$$u(-x,t) = \int_{-\infty}^{\infty} S(-x-y,t)\,\phi(y)\,dy$$

COV $z=-x$

$$= \int_{+\infty}^{-\infty} S(-x+z,t)\,\phi(-z)\,(-dz)$$

$$= \int_{-\infty}^{+\infty} S(-x+z,t)\,\phi(z)\,dz \quad \leftarrow \begin{cases}\text{Switching}\\ \text{limits and}\\ \text{using } \phi(-z)=\phi(z)\end{cases}$$

Finally, $S(-x+z,t) = \frac{1}{\sqrt{4\pi kt}}\, e^{-(-x+z)^2/4kt} = \frac{1}{\sqrt{4\pi kt}}\, e^{-(x-z)^2/4kt}$
$= S(x-z,t)$

$\therefore\ u(-x,t) = \int_{-\infty}^{\infty} S(x-z,t)\,\phi(z)\,dz = u(x,t)$ $\square$

(b) $u(-x,t) = u(x,t)$

$\Rightarrow -u_x(-x,t) = u_x(x,t)$ $\left\{\begin{array}{l}\text{Differentiating both sides}\\ \text{and using the chain rule}\end{array}\right.$

$\therefore$ for $x=0$, $-u_x(0,t) = u_x(0,t) \Rightarrow \underline{u_x(0,t) = 0}$ $\square$

Or, find $u_x(x,t) = \partial/\partial x \int_{-\infty}^{\infty} S(x-y,t)\,\phi(y)\,dy$

$$= \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} -\frac{(x-y)}{2kt}\, e^{-(x-y)^2/4kt}\, \phi(y)\,dy.$$

Then $u_x(0,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} y/2kt\; e^{-y^2/4kt}\, \phi(y)\,dy = \int_{-\infty}^{\infty} (\text{odd})(\text{even})(\text{even}) = 0$

5. [10 pts] Consider the wave equation on the finite interval [0,1] with the following initial conditions and boundary conditions:

$$
\begin{aligned}
&u_{tt} = c^2 u_{xx} && 0 < x < 1 \\
(IC) \rightarrow\ &u(x,0) = 0;\ u_t(x,0) = x && \\
&u_x(0,t) - u(0,t) = 0 && \Rightarrow\ u_x(0,t) = u(0,t) \\
&u_x(1,t) + u(1,t) = 0. && \Rightarrow\ u_x(1,t) = -u(1,t)
\end{aligned}
$$

Find the energy $E(t)$. (Hint: First, find $\frac{\mathrm{d}}{\mathrm{dt}}E(t)$. Then integrate in time from 0 to t.)

Either multiply the equation by u_t and integrate or . .

$$\frac{d}{dt}E(t) = \frac{d}{dt}\int_0^1 \left(u_t^2 + c^2 u_x^2\right)dx = \int_0^1 \left(2u_t u_{tt} + 2c^2 u_x u_{xt}\right)dx$$

$$= \int_0^1 2u_t u_{tt}\,dx - \int_0^1 2c^2 u_{xx} u_t\,dx + c^2 u_x u_t \Big|_0^1$$

$$= \int_0^1 2u_t \underbrace{(u_{tt} - c^2 u_{xx})}_{0}\,dx + c^2 u_x u_t \Big|_0^1$$

$$= c^2 u_x(1,t) u_t(1,t) - c^2 u_x(0,t) u_t(0,t)$$

using boundary conditions $\rightarrow$ $= -c^2 u(1,t) u_t(1,t) - c^2 u(0,t) u_t(0,t)$

$$= -\frac{c^2}{2}\,\partial/\partial t\left[(u(1,t))^2 + (u(0,t))^2\right]$$

Integrating both sides from 0 to t:

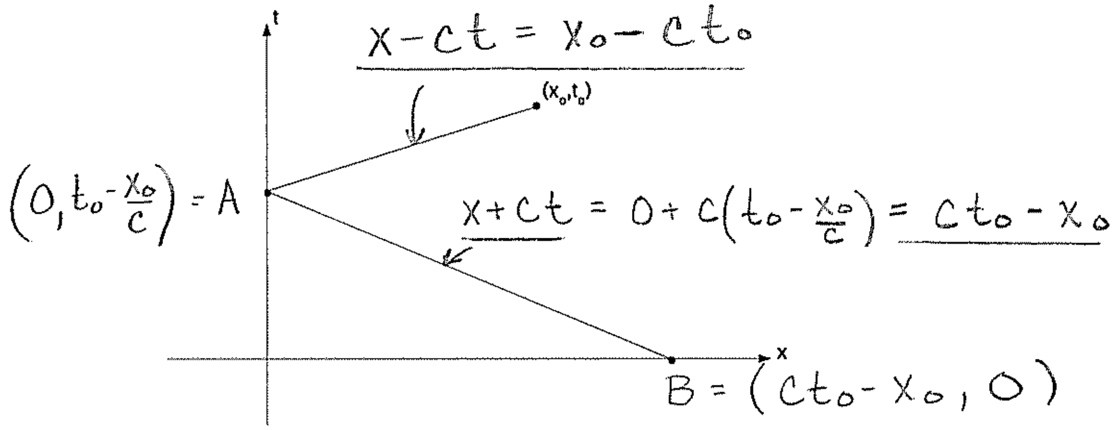
$$E(t) - E(0) = -\frac{c^2}{2}\left(u(1,t)^2 + u(0,t)^2\right) + \frac{c^2}{2}\underbrace{\left(u(1,0)^2 + u(0,0)^2\right)}_{=0 \text{ by } (IC)}$$

Also $E(0) = \int_0^1 \left(u_t(0,x)^2 + u_x(0,x)^2\right)dx$ with $u_x(0,x) = 0$

$$= \int_0^1 x^2\,dx = 1/3$$

$$\Rightarrow E(t) = 1/3 - \frac{c^2}{2}\left(u(1,t)^2 + u(0,t)^2\right)$$

6. [5 pts] Consider a point (x_o, t_o) as in the picture below (the point is in the first quadrant and satisfies $x_o < ct_o$). If the lines drawn are characteristics for the wave equation $u_{tt} = c^2 u_{xx}$, find the equation of each line and find the points where the lines intersect the axes.



At point $A = (x_1, t_1)$, $x_1 = 0$

$$0 - ct_1 = x_o - ct_o \Rightarrow t_1 = t_o - \frac{x_o}{c}$$

7. [10 pts] Solve

$$\begin{cases} u_{tt} - c^2 u_{xx} = xt^2 & \text{for } -\infty < x < \infty \\ u(x,0) = 0 \\ u_t(x,0) = e^{-x} \end{cases}$$

<u>(WE) with a source</u> $f(x,t) = xt^2$

$$u(x,t) = 0 + \frac{1}{2c}\int_{x-ct}^{x+ct} e^{-y}\,dy + \frac{1}{2c}\iint_\Delta f$$

$$= \frac{1}{2c}\left(-e^{-y}\right)\Big|_{x-ct}^{x+ct} + \frac{1}{2c}\iint_\Delta f$$

$$= \frac{1}{2c}\left(e^{-(x-ct)} - e^{-(x+ct)}\right) + \frac{1}{2c}\iint_\Delta f$$

$$= \frac{1}{2c}e^{-x}\left(e^{ct} - e^{-ct}\right) + \frac{1}{2c}\iint_\Delta f$$

$$= \frac{1}{c}e^{-x}\sinh(ct) + \frac{1}{2c}\iint_\Delta f.$$

$$\iint_\Delta f = \int_0^t \int_{x-c(t-s)}^{x+c(t-s)} ys^2\,dy\,ds = \int_0^t s^2\left(\frac{y^2}{2}\right)\Big|_{x-c(t-s)}^{x+c(t-s)} ds$$

$$= \int_0^t \frac{s^2}{2}\Big\{\left(\cancel{x^2} + 2cx(t-s) + \cancel{c^2(t-s)^2}\right)$$

$$- \left(\cancel{x^2} - 2cx(t-s) + \cancel{c^2(t-s)^2}\right)\Big\}\,ds$$

$$= \int_0^t 2cx\,s^2(t-s)\,ds = 2cx\left(\frac{ts^3}{3} - \frac{s^4}{4}\right)\Big|_0^t$$

$$= 2cx\left(t^4/3 - t^4/4\right) = 2cx\,t^4/12$$

$$\underline{u(x,t) = \frac{1}{c}e^{-x}\sinh(ct) + \frac{xt^4}{12}}$$

8. [10 pts] On the finite interval $[0,1]$, consider the problem

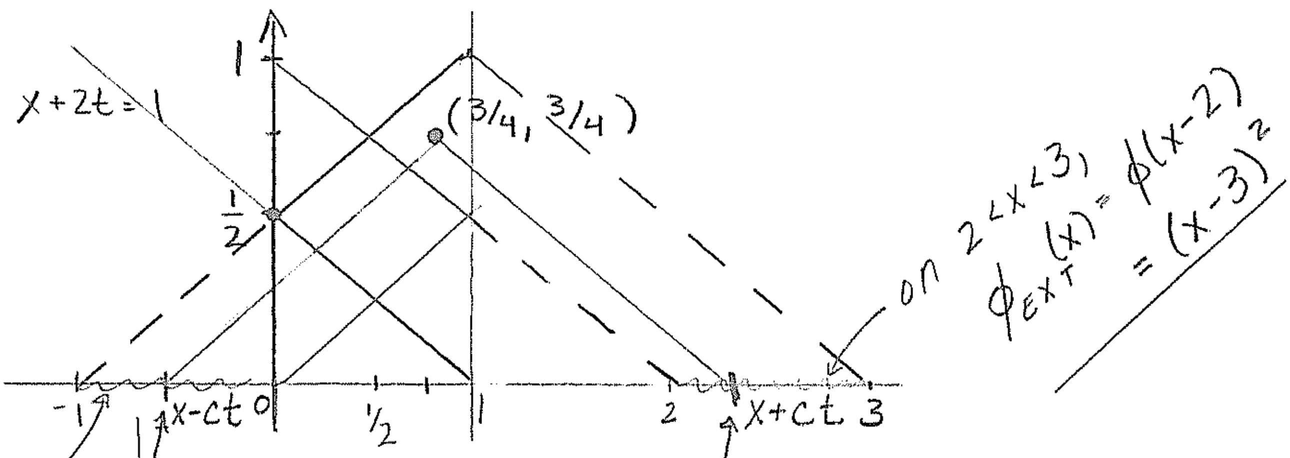
$$c=2 \qquad \begin{cases} u_{tt} = 4u_{xx} & \text{for } 0 < x < 1 \\ u(x,0) = (x-1)^2;\ u_t(x,0) = 1 & \\ u(0,t) = u(1,t) = 0. & \end{cases}$$

(a) Write the formulas for the extensions of the initial conditions that you would use to solve this problem.

(b) Find the solution $u(x,t)$ at the point $(x,t) = (\frac{3}{4}, \frac{3}{4})$.

(a) $$\phi_{EXT}(x) = \begin{cases} -(-x-1)^2 & \text{if } -1 < x < 0 \\ (x-1)^2 & \text{if } 0 < x < 1 \\ \text{periodic extension, of period 2} & \end{cases} \qquad \psi_{EXT}(x) = \begin{cases} -1 & -1 < x < 0 \\ +1 & 0 < x < 1 \\ \text{periodic extension.} & \end{cases}$$

(b)



here $\phi_{EXT}(x) = -(-x-1)^2$

$3/4 - 2(3/4) = -3/4$ $\qquad$ $3/4 + 2(3/4) = 9/4$

Using d'Alembert's formula:

$$u(\tfrac{3}{4}, \tfrac{3}{4}) = \frac{1}{2}\left[-(3/4-1)^2 + (9/4-3)^2\right] + \frac{1}{4}\int_{-3/4}^{9/4} \psi_{EXT}(y)\,dy$$

$$= \frac{1}{2}\left[-\left(\frac{1}{4}\right)^2 + \left(\frac{3}{4}\right)^2\right] + \frac{1}{4}\left\{\int_{-3/4}^{0}(-1)\,dy + \int_0^1 (+1)\,dy + \int_1^2 (-1)\,dy + \int_2^{9/4}(+1)\,dy\right\}$$

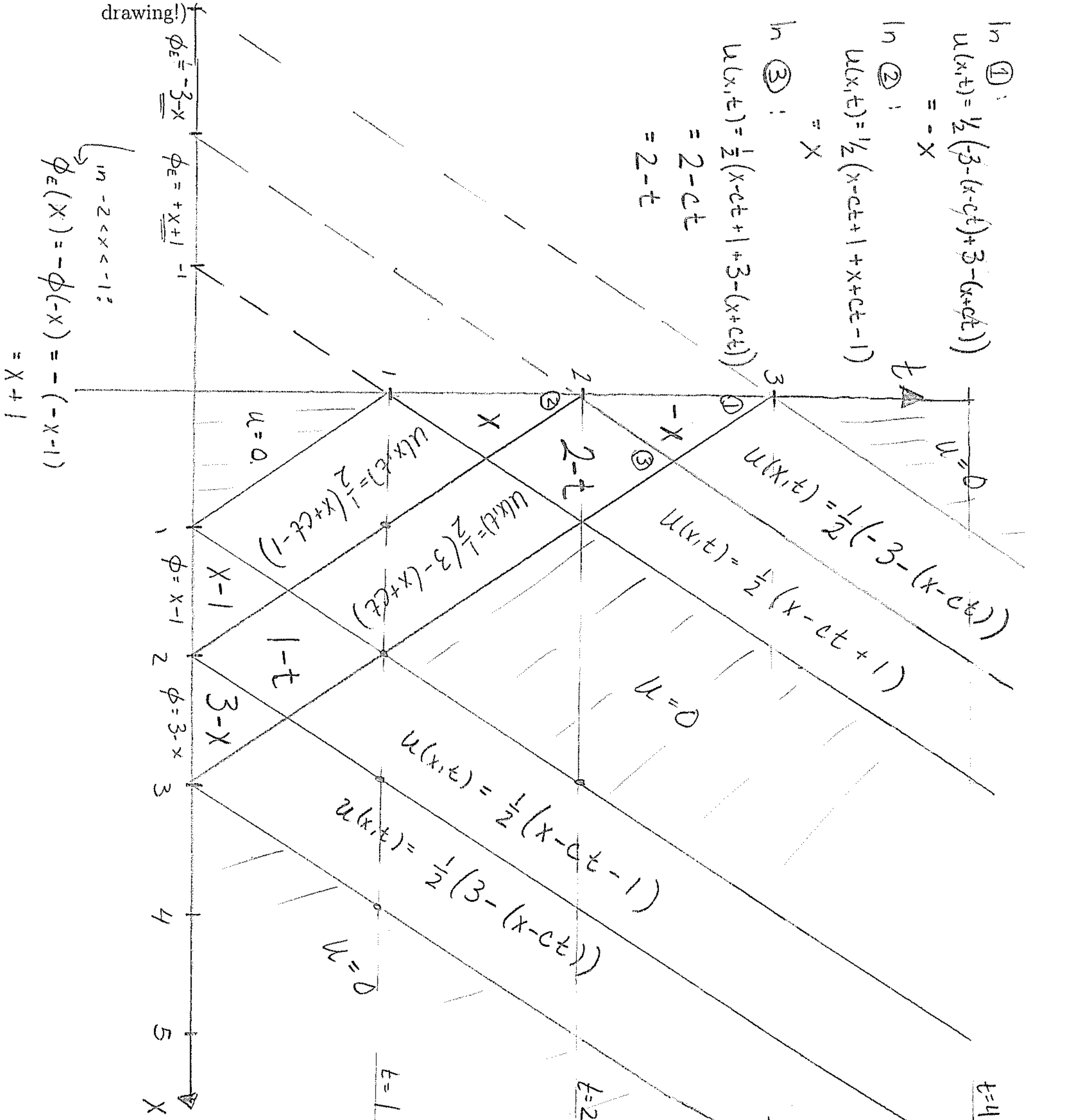
$$= \frac{9-1}{2\cdot 16} + \frac{1}{4}\left(-3/4 + \cancel{1} - \cancel{1} + (9/4 - 2)\right)$$

$$= \frac{1}{4} + \frac{1}{4}\left(-3/4 + 1/4\right) = \frac{1}{4} - \frac{1}{8} = \boxed{\frac{1}{8}}$$

9. [15 pts] Consider the following problem on the half line:

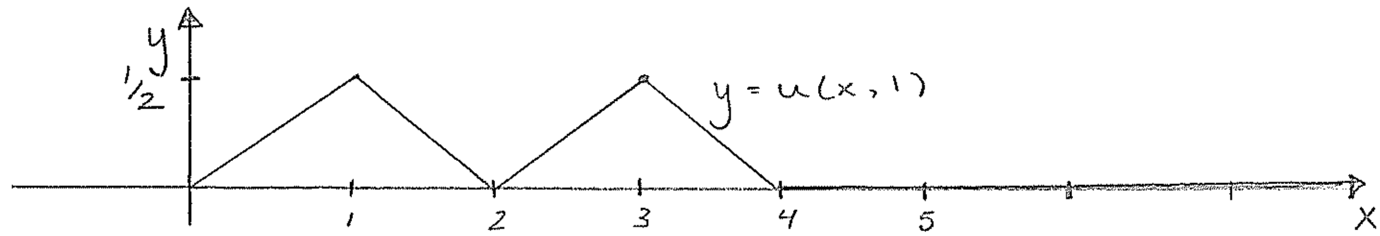
$$C=1 \qquad \begin{cases} u_{tt} = u_{xx} & \text{for } 0 < x < \infty \\ u(x,0) = \phi(x);\ u_t(x,0) = 0 \\ u(0,t) = 0 \end{cases} \qquad \text{where} \quad \phi(x) = \begin{cases} 0 & 0 \le x < 1 \\ x-1 & 1 \le x < 2 \\ 3-x & 2 \le x < 3 \\ 0 & 3 \le x. \end{cases}$$

Draw a large picture of the domain with the important characteristics and find the solution u in each region. Then, draw pictures of the solution $u(x,t)$ at the times $t = 1$, $t = 2$ and $t = 4$. (Label all the important points on these pictures. You also have the next page for drawing!)



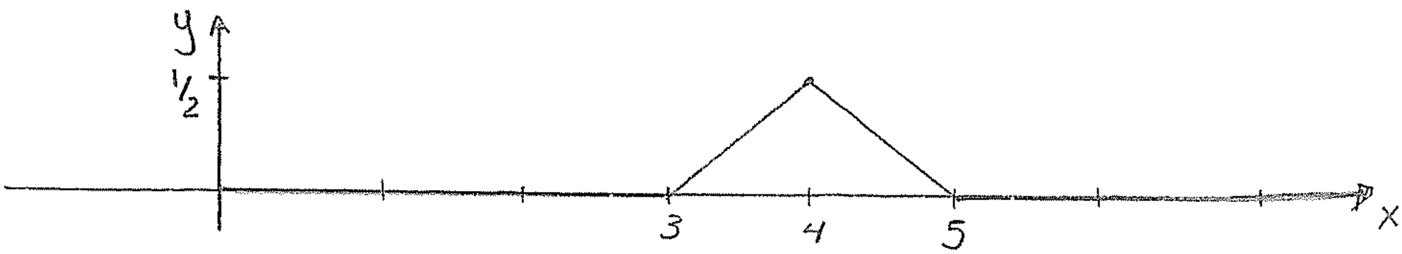
$\underline{t=1}$

$$u(x,1) = \begin{cases} \tfrac{1}{2}x & 0 < x < 1 \\ \tfrac{1}{2}(2-x) & 1 < x < 2 \\ \tfrac{1}{2}(x-2) & 2 < x < 3 \\ \tfrac{1}{2}(4-x) & 3 < x < 4 \\ 0 & x > 4 \end{cases}$$



$\underline{t=2}$

$$u(x,2) = \begin{cases} 0 & 0 < x < 3 \\ \tfrac{1}{2}(x-3) & 3 < x < 4 \\ \tfrac{1}{2}(5-x) & 4 < x < 5 \\ 0 & x > 5 \end{cases}$$



$\underline{t=4}$

$$u(x,4) = \begin{cases} 0 & 0 < x < 1 \\ \tfrac{1}{2}(1-x) & 1 < x < 2 \\ \tfrac{1}{2}(x-3) & 2 < x < 3 \\ 0 & 3 < x < 5 \\ \tfrac{1}{2}(x-5) & 5 < x < 6 \\ \tfrac{1}{2}(7-x) & 6 < x < 7 \\ 0 & x > 7 \end{cases}$$