Formulas

\[ S(x,t) = \frac{1}{\sqrt{4\pi kt}} e^{-\frac{x^2}{4kt}} \quad \text{and} \quad f_1f(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-p^2} \, dp \]

For any constant \( a \), the solution of

\[ \begin{cases} u_t = ku_{xx} & \text{for } -\infty < x < \infty (t > 0) \\ u(x,0) = ax^2 \end{cases} \]

is \( u(x,t) = ax^2 + 2akt \).

For any constant \( a \), the solution of

\[ \begin{cases} u_t = ku_{xx} & \text{for } -\infty < x < \infty (t > 0) \\ u(x,0) = e^{ax} \end{cases} \]

is \( u(x,t) = e^{ax + a^2kt} \).

For the wave equation on an interval \([a, b]\), the energy at time \( t \) is

\[ E(t) = \int_a^b \left( [u_t(x,t)]^2 + c^2[u_x(x,t)]^2 \right) \, dx \]
1. [15 pts] Solve $u_x - 2u_y - 5u = 2x^2 - 3xy - 2y^2$ with the condition $u(x, 0) = e^x$.

Let $x' = x - 2y$

$y' = 2x + y$

Then $u_x - 2u_y = (u_{x'} + 2y'y') - 2(-2u_x + yy')$

$= 5u_x'$

Also $2x^2 - 3xy - 2y^2 = (2x + y)(x - 2y) = y'x'$

Solve: $5u_{x'} - 5u = x'y'$

Integrating factor: $e^{-x'} \Rightarrow 5(e^{-x'}u' - e^{-x'}u) = e^{-x'}y'$

$\Rightarrow 5 \frac{\partial}{\partial x'} (e^{-x'}u) = e^{-x'}x'y'$

$\Rightarrow 5e^{-x'}u = y' \int x'e^{-x'}$

$= y'(-x'e^{-x'} - e^{-x'}) + f(y')$

(Check integral: $\frac{\partial}{\partial x'} (-x'e^{-x'} - e^{-x'}) = -x'e^{-x'} - e^{-x'} + e^{-x'}$

$= xe^{-x'} \checkmark$

Therefore,

$u = \frac{1}{5} y'(-x' - 1) + \frac{1}{5} e^{-x'}f(y')$

In terms of $x$ and $y$:

$u(x, y) = -\frac{1}{5} (2x+y)(x-2y+1) + \frac{1}{5} e^{x-2y}f(2x+y)$
Since \( u(x, 0) = e^x \)

\[
- \frac{1}{5} (2x)(1 + x) + \frac{1}{5} e^x \int (2x) = e^x
\]

\[
\frac{1}{5} e^x \int (2x) = e^x + \frac{1}{5} (2x + 2x^2)
\]

\[
f(2x) = 5 + e^{-x}(2x + 2x^2)
\]

\[
f(z) = 5 + e^{-2z/2}(z + \frac{z^2}{2})
\]
2. [10 pts] Solve for \( u \) (in terms of the error function).

\[
\begin{align*}
\phi(x) &= u(x, 0) = \begin{cases} 
 0 & x < 0 \\
 1 & x \geq 0 
\end{cases} \\
\frac{1}{4} u(t, x) - 4te^{-2x} &= \int_0^t \int_0^\infty S(x-y, t-s) \left[ 4se^{-2y} \right] dy 
\end{align*}
\]

\( (DE) \) with a source

\[
\begin{align*}
\phi(x) = u(x, 0) = \begin{cases} 
 0 & x < 0 \\
 1 & x \geq 0 
\end{cases}
\end{align*}
\]

Sol'n of \( (DE) \) at time \( t-s \) with IC \( e^{-2x} \)

\[
\begin{align*}
u(x, t) &= \int_0^\infty S(x-y, t) \phi(y) dy + \int_0^t \int_0^\infty S(x-y, t-s) \left[ 4se^{-2y} \right] dy ds \\
&= \int_0^\infty \frac{e^{-(x-y)^2/4}}{\sqrt{\pi t}} y dy + \int_0^t 4s e^{-2x + (t-s)} ds \\
&= \int_0^\infty \frac{e^{-(x-y)^2/4}}{\sqrt{\pi t}} y dy + \int_0^t 4s e^{-2x + (t-s)} ds \\
&= \frac{2}{\sqrt{\pi}} \left[ \int_0^\infty e^{-p^2} dp + \frac{x}{\sqrt{\pi}} \right] - \frac{1}{\sqrt{\pi}} \left[ \int_0^\infty pe^{-p^2} dp + 4e^{-2x+t} \left[ \int_0^t e^{-s} ds - se^{-s} \right] \right] \\
&= \frac{2}{\sqrt{\pi}} \left[ \frac{x}{\sqrt{\pi}} \right] + \frac{x}{2} \mathcal{E}_n \left( \frac{x}{\sqrt{t}} \right) + \frac{\sqrt{t}}{2\sqrt{\pi}} e^{-x^2/4} + 4e^{-2x} \left[ -1 + e^{t-t} \right] \\
u(x, t) &= \frac{x}{2} + \frac{x}{2} \mathcal{E}_n \left( \frac{x}{\sqrt{t}} \right) + \frac{\sqrt{t}}{2\sqrt{\pi}} e^{-x^2/4} + 4e^{-2x} \left[ -1 + e^{t-t} \right]
\end{align*}
\]
3. [5+5 pts] Consider the following problem for $u$:

$$
\begin{cases}
  u_t - ku_{xx} + xe^{-t} = 0 \\
  u(x, 0) = \delta(x - 2) + x^2 + x \\
  u_x(0, t) = e^{-t}.
\end{cases}
$$

Let $v(x, t) = u(x, t) - xe^{-t}$.

(a) What problem does $v$ solve?

(b) [5 pts] Find the solution $u$.

(a) Compute:

$$
\begin{align*}
  v_t &= u_t + xe^{-t} \\
  v_x &= u_x - e^{-t} \\
  v_{xx} &= u_{xx}
\end{align*}
$$

$v$ solves

$$
\begin{cases}
  v_t - kv_{xx} = u_t + xe^{-t} - u_{xx} = 0 & 0 < x < \infty \\
  v(x, 0) = u(x, 0) - xe^0 = \delta(x - 2) + x^2 \\
  v_x(0, t) = u_x(0, t) - e^{-t} = 0
\end{cases}
$$

(b) $v$ solves (DE) on the half-line with the Neumann BC $v_x(0, t) = 0$, so we can use an even extension of the IC $\phi(x) = \delta(x - 2) + x^2$ to solve for $v$

$$
\phi_{\text{Even}}(x) = \begin{cases} 
  \delta(x - 2) + x^2 & x > 0 \\
  \delta(-x - 2) + x^2 & x < 0
\end{cases}
$$

Therefore,

$$
\begin{align*}
  v(x, t) &= \int_{-\infty}^{0} S(x - y, t)[\delta(y - 2) + y^2]dy + \int_{0}^{\infty} S(x - y, t)[\delta(y - 2) + y^2]dy \\
  &= S(x + 2, t) + S(x - 2, t) + \int_{-\infty}^{\infty} S(x - y, t)y^2dy \\
  &= S(x + 2, t) + S(x - 2, t) + x^2 + 2kt
\end{align*}
$$

$\Rightarrow$ $u(x, t) = \frac{1}{\sqrt{4\pi kt}} \left[ e^{-(x + 2)^2/4kt} + e^{-(x - 2)^2/4kt} \right] + x^2 + 2kt + x e^{-t}$
4. [10 pts] Consider the heat equation on the whole line with an even initial condition:

\[ u_t = k u_{xx}, \quad -\infty < x < \infty \]
\[ u(x, 0) = \phi(x), \]

where \( \phi \) is an even function.

(a) Prove that the solution \( u(x, t) \) is an even function of \( x \).

(b) Prove that \( u_x(0, t) = 0 \).

\[(a) \quad u(x, t) = \int_{-\infty}^{\infty} S(x-y, t) \phi(y) \, dy \]

We want to show that \( u(-x, t) = u(x, t) \)

\[ u(-x, t) = \int_{-\infty}^{\infty} S(-x-y, t) \phi(y) \, dy \]

\[ = \int_{-\infty}^{\infty} S(-x+z, t) \phi(-z) (-dz) \]

Switching limits and using \( \phi(-z) = \phi(z) \)

Finally, \( S(-x+z, t) = \frac{1}{\sqrt{4\pi kt}} e^{-\frac{(x-z)^2}{4kt}} = \frac{1}{\sqrt{4\pi kt}} e^{-\frac{(x-2z)^2}{4kt}} \)

\[ = S(x-2z, t) \]

\[ \Rightarrow \quad u(-x, t) = \int_{-\infty}^{\infty} S(x-2z, t) \phi(z) \, dz = u(x, t) \]

\[(b) \quad u(-x, t) = u(x, t) \]

\[ \Rightarrow -u_x(-x, t) = u_x(x, t) \]

Differentiating both sides and using the chain rule

\[ \Rightarrow \quad \text{for } x = 0, \quad -u_x(0, t) = u_x(0, t) \Rightarrow u_x(0, t) = 0 \]

\[ \partial_t \text{, find } \partial_x \text{, and } \]

\[ u_x(x, t) = \partial_x \int_{-\infty}^{\infty} S(x-y, t) \phi(y) \, dy \]

\[ = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} \frac{-(x-y)}{2kt} e^{-\frac{(x-y)^2}{4kt}} \phi(y) \, dy \]

Then \( u_x(0, t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} y \frac{1}{2kt} e^{-\frac{y^2}{4kt}} \phi(y) \, dy = \int_{-\infty}^{\infty} \left( \text{odd} \right) \left( \text{even} \right) \left( \text{even} \right) = 0 \)
5. [10 pts] Consider the wave equation on the finite interval $[0,1]$ with the following initial conditions and boundary conditions:

$$u_{tt} = c^2 u_{xx} \quad \text{for} \quad 0 < x < 1$$

$$(I) \rightarrow u(x,0) = 0; \quad u_t(x,0) = x$$

$$(C) \rightarrow u_x(0,t) - u(0,t) = 0 \quad \Rightarrow \quad u_x(0,t) = u(0,t)$$

$$u_x(1,t) + u(1,t) = 0. \quad \Rightarrow \quad u_x(1,t) = -u(1,t)$$

Find the energy $E(t)$. (Hint: First, find $\frac{d}{dt} E(t)$ Then integrate in time from 0 to $t$.)

Either multiply the equation by $u_t$ and integrate

$$\frac{d}{dt} E(t) = \frac{d}{dt} \int_0^1 \left( u_t^2 + c^2 u_x^2 \right) dx = \int_0^1 \left( 2u_t u_{tt} + 2c^2 u_x u_{xt} \right) dx$$

$$= \left[ \int_0^1 2u_t u_{tt} dx - \int_0^1 2c^2 u_x u_t dx + c^2 u_x u_t \right]_0^1$$

$$= \int_0^1 2u_t (u_{tt} - c^2 u_{xx}) dx + c^2 u_x u_t \bigg|_0^1$$

$$= c^2 u_x(1,t) u_t(1,t) - c^2 u_x(0,t) u_t(0,t)$$

Using boundary conditions:

$$= -c^2 u(1,t) u_t(1,t) - c^2 u(0,t) u_t(0,t)$$

Integrating both sides from 0 to $t$:

$$E(t) - E(0) = -\frac{c^2}{2} \left( u(1,t)^2 + u(0,t)^2 \right) + \frac{c^2}{2} \left( u(1,0)^2 + u(0,0)^2 \right)$$

Also:

$$E(0) = \int_0^1 \left( u_t(0,x)^2 + u_x(0,x)^2 \right) dx$$

$$= \int_0^1 x^2 dx = \frac{1}{3}$$

$$\Rightarrow E(t) = \frac{1}{3} - \frac{c^2}{2} \left( u(1,t)^2 + u(0,t)^2 \right)$$
6. [5 pts] Consider a point \((x_0, t_0)\) as in the picture below (the point is in the first quadrant and satisfies \(x_0 < ct_0\)). If the lines drawn are characteristics for the wave equation \(u_{tt} = c^2 u_{xx}\), find the equation of each line and find the points where the lines intersect the axes.

\[
\begin{align*}
A &= (0, t_0 - \frac{x_0}{c}) \\
\text{(0, } t_0 - \frac{x_0}{c}) &= A \hspace{2cm} \frac{x - ct}{t_0 - \frac{x_0}{c}} = \frac{x_0 - ct_0}{t_0 - \frac{x_0}{c}} \\
B &= (ct_0 - x_0, 0) \hspace{2cm} \frac{x + ct}{c(t_0 - \frac{x_0}{c})} = \frac{ct_0 - x_0}{c}
\end{align*}
\]

At point \(A = (x_1, t_1)\), \(x_1 = 0\)

\[
0 - ct_1 = x_0 - ct_0 \Rightarrow t_1 = t_0 - \frac{x_0}{c}
\]
7. [10 pts] Solve
\[
\begin{aligned}
&\begin{cases}
u_{tt} - c^2 u_{xx} = xt^2 & \text{for } -\infty < x < \infty \\
u(x,0) = 0 \\
u_t(x,0) = e^{-x}
\end{cases}
\end{aligned}
\]

\[
(WE) \quad \text{with a source} \quad f(x,t) = xt^2
\]

\[
u(x,t) = 0 + \frac{1}{2c} \int_{x-ct}^{x+ct} e^{-y} dy + \frac{1}{2c} \iint_\Delta f
\]

\[
= \frac{1}{2c} \left( - e^{-y} \right) \bigg|_{x-ct}^{x+ct} + \frac{1}{2c} \iiint_\Delta f
\]

\[
= \frac{1}{2c} \left( e^{ct} - e^{-(x+ct)} \right) + \frac{1}{2c} \iiint_\Delta f
\]

\[
= \frac{1}{2c} e^{-x} (e^{ct} - e^{ct}) + \frac{1}{2c} \iiint_\Delta f
\]

\[
= \frac{1}{c} e^{-x} \sinh (ct) + \frac{1}{2c} \iiint_\Delta f.
\]

\[
\iiint_\Delta f = \int_0^t \int_{x-ct}^{x+ct} y s^2 dy ds = \int_0^t \int_{x-ct}^{x+ct} \left( y^2 \right) ds
\]

\[
= \int_0^t \int_{x-ct}^{x+ct} \left( x^2 + 2cx(t-s) + c^2(t-s)^2 \right) - \left( x^2 - 2cx(t-s) + c^2(t-s)^2 \right) ds
\]

\[
= \int_0^t 2c x s^2 (t-s) ds = 2c x \left( \frac{t s^3}{3} - \frac{s^4}{4} \right) \bigg|_0^t
\]

\[
= 2c x \left( \frac{t^4}{3} - \frac{t^4}{4} \right) = 2c x \frac{t^4}{12}
\]

\[
u(x,t) = \frac{1}{c} e^{-x} \sinh (ct) + \frac{xt^4}{12}.
\]
8. [10 pts] On the finite interval \([0, 1]\), consider the problem

\[
\begin{cases} 
  u_t = 4u_{xx} \\
  u(x, 0) = (x - 1)^2; \quad u_t(x, 0) = 1 \\
  u(0, t) = u(1, t) = 0.
\end{cases}
\]

(a) Write the formulas for the extensions of the initial conditions that you would use to solve this problem.

(b) Find the solution \(u(x, t)\) at the point \((x, t) = (\frac{3}{4}, \frac{3}{4})\).

(a) \(\phi_{\text{ext}}(x) = \begin{cases} 
  -(-x-1)^2 & \text{if } -1 < x < 0 \\
  (x-1)^2 & \text{if } 0 < x < 1 \\
  \text{periodic extension,} \\
  \text{of period 2}
\end{cases}\)

(b) Using d'Alembert's formula:

\[
U\left(\frac{3}{4}, \frac{3}{4}\right) = \frac{1}{2} \left[ -\left(\frac{3}{4} - 1\right)^2 + \left(\frac{9}{4} - 3\right)^2 \right] + \frac{1}{4} \int_{-\frac{3}{4}}^{\frac{3}{4}} 2\phi_{\text{ext}}(y)\,dy
\]

\[
= \frac{1}{2} \left[ -\left(\frac{1}{4}\right)^2 + \left(\frac{3}{4}\right)^2 \right] + \frac{1}{4} \left[ \int_{-\frac{3}{4}}^{0} (-1)\,dy + \int_{0}^{\frac{1}{4}} (+1)\,dy + \int_{\frac{1}{4}}^{\frac{3}{4}} (-1)\,dy + \int_{\frac{3}{4}}^{\frac{9}{4}} (+1)\,dy \right]
\]

\[
= \frac{9}{2} - \frac{1}{16} + \frac{1}{4} \left[ -3\frac{3}{4} + 1 - 1 + (\frac{9}{4} - 2) \right]
\]

\[
= \frac{1}{4} + \frac{1}{4} \left( -3\frac{3}{4} + \frac{1}{4} \right) = \frac{1}{4} - \frac{1}{8} = \boxed{\frac{1}{8}}
\]
9. [15 pts] Consider the following problem on the half line:

\[
\begin{aligned}
C=1 \\
\begin{cases}
  u_{tt} = u_{xx} & \text{for } 0 < x < \infty \\
u(x,0) = \phi(x); u_t(x,0) = 0 & \text{where } \phi(x) = \begin{cases}
  0 & 0 \leq x < 1 \\
x - 1 & 1 \leq x < 2 \\
3 - x & 2 \leq x < 3 \\
0 & 3 \leq x.
\end{cases}
\end{cases}
\end{aligned}
\]

Draw a large picture of the domain with the important characteristics and find the solution \( u \) in each region. Then, draw pictures of the solution \( u(x,t) \) at the times \( t = 1 \), \( t = 2 \) and \( t = 4 \). (Label all the important points on these pictures. You also have the next page for drawing.)
\[ t=1 \quad u(x,1) = \begin{cases} 
\frac{1}{2}x & 0 < x < 1 \\
\frac{1}{2}(2-x) & 1 < x < 2 \\
\frac{1}{2}(x-2) & 2 < x < 3 \\
\frac{1}{2}(4-x) & 3 < x < 4 \\
0 & x > 4 
\end{cases} \]

\[ t=2 \quad u(x,2) = \begin{cases} 
0 & 0 < x < 3 \\
\frac{1}{2}(x-3) & 3 < x < 4 \\
\frac{1}{2}(5-x) & 4 < x < 5 \\
0 & x > 5 
\end{cases} \]

\[ t=4 \quad u(x,4) = \begin{cases} 
0 & 0 < x < 1 \\
\frac{1}{2}(1-x) & 1 < x < 2 \\
\frac{1}{2}(x-3) & 2 < x < 3 \\
0 & 3 < x < 5 \\
\frac{1}{2}(x-5) & 5 < x < 6 \\
\frac{1}{2}(7-x) & 6 < x < 7 \\
0 & x > 7 
\end{cases} \]