

Math 124A: PDEs

Solving the diffusion (heat) equation

Fix a positive number k and consider the diffusion equation on the whole line

$$(DE) \quad u_t = ku_{xx} \quad (\text{on } 0 < t < \infty, -\infty < x < \infty)$$

Properties of the diffusion equation:

- If $S_1(x, t)$ and $S_2(x, t)$ are both solutions of (DE) (so $(S_1)_t = k(S_1)_{xx}$ and $(S_2)_t = k(S_2)_{xx}$), then the function $u(x, t) = a_1S_1(x, t) + a_2S_2(x, t)$ (where a_1 and a_2 are any two numbers) is a solution of (DE).

You know this because the diffusion equation is linear! You can also check it by taking partial derivatives: Find

$$\begin{aligned} u_t &= \frac{\partial u}{\partial t} = \frac{\partial}{\partial t} (a_1S_1 + a_2S_2) \\ u_{xx} &= \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2}{\partial x^2} (a_1S_1 + a_2S_2) \end{aligned}$$

Check that $u_t - ku_{xx} = 0$!

- If $S(x, t)$ is a solution of (DE), then the function $u(x, t) = S(x - y, t)$ is a solution of (DE) (where y is any fixed number!)

Check:

$$\begin{aligned} u_t(x, t) &= [S(x - y, t)]_t = S_t(x - y, t) \\ u_{xx}(x, t) &= [S(x - y, t)]_{xx} = S_{xx}(x - y, t). \end{aligned}$$

Then, $u_t(x, t) - ku_{xx}(x, t) = S_t(x - y, t) - kS_{xx}(x - y, t) = 0$ since S solves (DE).

- If $S(x, t)$ is a solution of (DE), then $u(x, t) = \int_{-\infty}^{\infty} S(x - y, t)g(y)dy$ is a solution of (DE) (for any function g of one variable).

This is also easy to check simply by differentiating u ! (If you can pull derivatives under the integral sign.) Intuitively, we can think of this integral as an infinite sum. In the integral above, we are summing solutions of (DE), so by

linearity, u should solve (DE). More rigorously, if we partition the line into intervals of size $\Delta y = \frac{1}{n}$, for each integer k , $\frac{1}{n}g(\frac{k}{n})$ is just some number, and by the previous property $(g(\frac{k}{n})\Delta y) S(x - \frac{k}{n}, t)$ solves (DE). By linearity,

$$S_n(x, t) = \sum_{k=-\infty}^{\infty} g\left(\frac{k}{n}\right) S\left(x - \frac{k}{n}, t\right) \Delta y$$

is a solution of (DE). Of course, as n gets large, $S_n(x, t) \rightarrow u(x, t)$! This implies that u also solves (DE) (to see this, take limits of both sides of the equation $(S_n)_t = k(S_n)_{xx}$).

- Finally, if $v(x, t)$ is a solution of (DE), the new function $u(x, t) = v(\sqrt{ax}, at)$ is also a solution of (DE) (for any positive number a)!

Check by differentiating (use the chain rule):

$$u_t(x, t) = \frac{\partial}{\partial t}[v(\sqrt{ax}, at)] = \left(\frac{\partial v}{\partial t}\right)(\sqrt{ax}, at) \frac{\partial}{\partial t}(at) = a v_t(\sqrt{ax}, at)$$

$$u_{xx}(x, t) = \frac{\partial}{\partial x}\left(\sqrt{a} \frac{\partial v}{\partial x}\right) = a v_{xx}(\sqrt{ax}, at).$$

Then, $u_t - ku_{xx} = 0$ at every point (with $t > 0$).

We will start by solving (DE) with special choice for the initial condition: The problem $(\star\star)$ will be

$$(\star\star) \begin{cases} S_t = kS_{xx} & (0 < t < \infty, -\infty < x < \infty) \\ S(x, 0) = \delta(x) & (-\infty < x < \infty). \end{cases}$$

The solution $S(x, t)$ that we find is alternately called the source function for the diffusion equation, Green's function for the diffusion equation, or the fundamental solution of the diffusion equation.

This solution gives us all others! If we want to solve the problem (\star)

$$(\star) \begin{cases} u_t = ku_{xx} & (0 < t < \infty, -\infty < x < \infty) \\ u(x, 0) = \phi(x) & (-\infty < x < \infty), \end{cases}$$

the solution is

$$u(x, t) = \int_{-\infty}^{\infty} S(x - y, t)\phi(y)dy.$$

Why? Well, by our properties of the diffusion equation, this solves (DE) (i.e., $u_t = ku_{xx}$). Also, we can check whether the initial condition holds:

$$u(x, 0) = \int_{-\infty}^{\infty} S(x - y, 0)\phi(y)dy = \int_{-\infty}^{\infty} \delta(x - y)\phi(y)dy = \phi(x).$$

Since $u(x, t)$ solves (DE) and $u(x, 0) = \phi(x)$, the function u solves the problem (\star). *Intuitively, this is true by linearity of the diffusion equation: We can write $\phi(x)$ as the integral of the functions $\phi(y)\delta(x-y)$, and since the equation is linear, the solution u should be the integral of the solutions $\phi(y)S(x-y, t)$, each of which has initial condition $\phi(y)\delta(x-y)$.*

All we have to do is figure out what the function $S(x, t)$ is! Instead, it will be easier to solve for Q where $S = \frac{\partial Q}{\partial x}$ and Q solves the problem

$$\begin{cases} Q_t = kQ_{xx} & (0 < t < \infty, -\infty < x < \infty) \\ Q(x, 0) = H(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases} \end{cases}$$

($S = \frac{\partial Q}{\partial x}$ will solve (DE) with $S(x, 0) = H'(x) = \delta(x)$!) Again, from our properties of the diffusion equation we know that $Q(\sqrt{ax}, at)$ solves (DE). But this function has initial condition $Q(\sqrt{ax}, 0) = H(\sqrt{ax}) = H(x)$! (Check that $H(\sqrt{ax}) = H(x)$.) Because of this special property of H , we expect that

$$Q(x, t) = Q(\sqrt{ax}, at)$$

since we expect that two solutions of (DE) that start off with same initial condition are the same (in other words, we expect uniqueness!) Then, the value of the function Q depends only on the ratio $\frac{x}{\sqrt{t}}$. ($Q(x, t)$ will be constant on the curves $x = c\sqrt{t}$!) This means that

$$Q(x, t) = g\left(\frac{x}{\sqrt{t}}\right)$$

for some function g of one variable (later, we'll call the variable z). We know Q solves (DE), and we will use that to figure out what ODE g solves. Start differentiating Q ... remember to use the chain rule! For the derivative with respect to t , hold x constant, and notice $\left(\frac{1}{\sqrt{t}}\right)_t = -\frac{1}{2t^{3/2}}$, so

$$Q_t(x, t) = g'\left(\frac{x}{\sqrt{t}}\right) \frac{\partial}{\partial t} \left(\frac{x}{\sqrt{t}}\right) = -\frac{x}{2t^{3/2}} g'\left(\frac{x}{\sqrt{t}}\right).$$

Also,

$$Q_x(x, t) = \frac{1}{\sqrt{t}} g'\left(\frac{x}{\sqrt{t}}\right) \quad \text{and} \quad Q_{xx}(x, t) = \frac{1}{t} g''\left(\frac{x}{\sqrt{t}}\right).$$

We know $Q_t = kQ_{xx}$. Plugging in our computations above, this means that

$$-\frac{x}{2t^{3/2}} g'\left(\frac{x}{\sqrt{t}}\right) = k \frac{1}{t} g''\left(\frac{x}{\sqrt{t}}\right)$$

Multiplying both sides by $\frac{t}{k}$:

$$-\frac{x}{2k\sqrt{t}}g'\left(\frac{x}{\sqrt{t}}\right) = g''\left(\frac{x}{\sqrt{t}}\right).$$

If we let $z = \frac{x}{\sqrt{t}}$, we have $-\frac{z}{2k}g'(z) = g''(z)$, or

$$g''(z) + \frac{z}{2k}g'(z) = 0.$$

We know how to solve this for g ... use an integrating factor! The integrating factor is $e^{\frac{z^2}{4k}}$, and we can calculate

$$\frac{d}{dz}\left(e^{\frac{z^2}{4k}}g'(z)\right) = e^{\frac{z^2}{4k}}g''(z) + \frac{2z}{4k}e^{\frac{z^2}{4k}}g'(z) = e^{\frac{z^2}{4k}}\left(g''(z) + \frac{z}{2k}g'(z)\right) = 0.$$

Therefore, there is some constant c such that

$$e^{\frac{z^2}{4k}}g'(z) = c.$$

We have found that $g'(z) = ce^{-\frac{z^2}{4k}}$. Now, we could integrate again to find $Q(x, t) = g(z)$ and use the initial conditions to figure out the constants... but all we really want to find is $S = \frac{\partial Q}{\partial x}$! Since $Q(x, t) = g(z)$ and $z = \frac{x}{\sqrt{t}}$,

$$S(x, t) = Q_x(x, t) = g'(z)z_x = \frac{c}{\sqrt{t}}e^{-\frac{x^2}{4kt}}.$$

The only thing we need to figure out is what the constant c is. We use the fact that as $t \rightarrow 0$, S should get close to its initial condition: $S(x, t) \rightarrow S(x, 0) = \delta(x)$. This means that $\int_{-\infty}^{\infty} S(x, t)dx \rightarrow \int_{-\infty}^{\infty} \delta(x)dx = 1$! We know how to integrate this (by changing variables from x to $p = x/\sqrt{4kt}$):

$$\int_{-\infty}^{\infty} \frac{c}{\sqrt{t}}e^{-\frac{x^2}{4kt}}dx = \frac{c}{\sqrt{t}} \int_{-\infty}^{\infty} e^{-p^2} \left(\sqrt{4kt} dp\right) = c\sqrt{4k} \int_{-\infty}^{\infty} e^{-p^2} dp = c\sqrt{4k}\sqrt{\pi}.$$

To make sure that this approaches 1, we need the constant $c = \frac{1}{\sqrt{4\pi k}}$.

Now, we know exactly what the source function is!

$$\boxed{S(x, t) = \frac{1}{\sqrt{4\pi kt}}e^{-\frac{x^2}{4kt}}.}$$

To solve the problem (\star) ((DE) with (IC) $u(x, 0) = \phi(x)$), we find u by integrating:

$$\boxed{u(x, t) = \int_{-\infty}^{\infty} S(x - y, t)\phi(y)dy = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} \phi(y)dy.}$$

You can take the formula above and check that it works! (Find $u_t - ku_{xx}$; it should work out to be 0. For the initial condition, we can't really plug in $t = 0$ — this give us the fraction “ $\frac{0}{0}$ ” — but we can take the limit as $t \rightarrow 0$! It's a fair amount of work, but we could prove that $u(x, t) \rightarrow \phi(x)$.)