**Math 124a: PDEs**

**The Wave Equation with a Source**

Consider the problem of the wave equation with a source:

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x,t) & \text{on } -\infty < x < \infty \\ u(x,0) = \phi(x); \quad u_t(x,0) = \psi(x). \end{cases}$$

The solution $u$ is given by the formula

$$\boxed{u(x,t) = \frac{1}{2}(\phi(x+ct) + \phi(x-ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)\,dy + \frac{1}{2c}\iint_\Delta f} \tag{1}$$

where the last term is the double integral of $f$ over the domain of dependence for the point $(x,t)$:

$$\frac{1}{2c}\iint_\Delta f = \frac{1}{2c}\int_0^t \left[\int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\,dy\right] ds.$$

Notice that we could check directly that this formula satisfies $u_{tt} - c^2 u_{xx} = f(x,t)$ (using the fundamental theorem of calculus carefully!) If $u$ is a solution, it is also easy to show that the formula must hold by calculating the integral

$$\iint_\Delta f = \iint_\Delta (u_{tt} - c^2 u_{xx}) = u(x,t) - \frac{1}{2}(\phi(x+ct) - \phi(x-ct)) - \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)\,dy.$$

This can be done directly (by changing variables to the characteristic coordinates $\xi$ and $\eta$) or by using Green's theorem. See §3.4 for details.

For the wave equation with a source on the half-line, we will consider the case of a Dirichlet boundary condition at $x = 0$. If we have the boundary condition $u(0,t) = 0$, we can use the formula for the solution on the whole line, taking the odd extensions of the initial conditions $\phi$ and $\psi$ as well as the odd extension (in the $x-$variable) of the source $f$. This works since, if all of these functions are odd, every term in the formula (1) equals 0 when $x = 0$. By $f_{\text{ODD}}$, the odd extension of $f$ in the $x-$variable, we mean

$$f_{\text{ODD}}(x,t) = \begin{cases} f(x,t) & \text{if } x > 0 \\ -f(-x,t) & \text{if } x < 0. \end{cases}$$

Given a point $(x,t)$, the domain of dependence is entirely in the first quadrant if $x - ct > 0$, so in this region, the boundary condition does not affect the solution. But in the region $x < ct$, the boundary condition has an effect, and the odd extensions of $\phi$, $\psi$, and $f$ can be used in the formula (1) to determine the solution.

Now, we wish to consider a general Dirichlet boundary condition. In other words, we want to solve the problem

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x,t) & \text{on } 0 < x < \infty \\ u(x,0) = \phi(x); \quad u_t(x,0) = \psi(x). \\ u(0,t) = h(t) \end{cases}$$

We will prove that the formula for the solution is given by

$$\boxed{\begin{array}{ll} u(x,t) = \frac{1}{2}(\phi(x+ct) + \phi(x-ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)\,dy + \frac{1}{2c}\iint_{\Delta} f & \text{if } x > ct \\ u(x,t) = \frac{1}{2}(\phi_{\text{ODD}}(x+ct) + \phi_{\text{ODD}}(x-ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{\text{ODD}}(y)\,dy & \text{if } x < ct \\ \qquad + \frac{1}{2c}\iint_{\Delta} f_{\text{ODD}} + h(t - \frac{x}{c}) & \end{array}} \tag{2}$$

Notice that in the region $x > ct$, the formula is identical to the solution for the problem on the whole line. The only effect of changing the boundary condition from 0 to $h(t)$ is to add a term that depends on $h$ to the solution in the region $x < ct$. The function $h(t - \frac{x}{c})$ is constant along the characteristics with positive slope, so this function is basically a wave propogating to the right.

To solve this problem on the half-line with the general Dirichlet boundary condition $u(0,t) = h(t)$, we will start by letting $v(x,t) = u(x,t) - h(t)$. For simplicity, let's assume $\phi = \psi = 0$. Then, $v$ solves the problem

$$\begin{cases} v_{tt} - c^2 v_{xx} = f(x,t) - h''(t) & \text{on } 0 < x < \infty \\ v(x,0) = -h(0); \; v_t(x,0) = -h'(0) \\ v(0,t) = 0, \end{cases}$$

and we know how to solve for $v$! If $x > ct$, we have

$$\begin{aligned} v(x,t) &= \frac{1}{2}(-h(0) - h(0)) + \frac{1}{2c}\int_{x-ct}^{x+ct} (-h'(0))\, dy + \frac{1}{2c}\iint_\Delta (f - h'') \\ &= -h(0) - th'(0) + \frac{1}{2c}\iint_\Delta h'' + \frac{1}{2c}\iint_\Delta f. \end{aligned}$$

We can calculate the double integral of $h''$ by first doing the integral with respect to $y$ and then integrating by parts:

$$\begin{aligned} -\frac{1}{2c}\iint_\Delta h'' &= -\int_0^t \left[\frac{1}{2c}\int_{x-ct}^{x+ct} 1\, dy\right] h''(s)\, ds = -\int_0^t h''(s)(t-s)\, ds \\ &= +\int_0^t (-1)h'(s)\, ds - h'(s)(t-s)\Big|_{s=0}^t \\ &= -h(t) + h(0) + th'(0) \end{aligned}$$

Plugging this into the formula for $v$, we have $v(x,t) = -h(t) + \frac{1}{2c}\iint_\Delta f$. Therefore, for $x > ct$,

$$u(x,t) = \frac{1}{2c}\iint_\Delta f.$$

As expected for this region, the boundary condition has no effect.

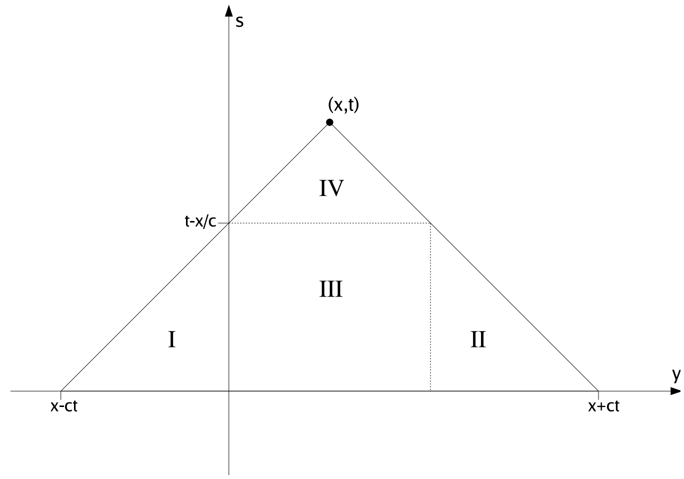
For the second region, $x < ct$, we have that

$$v(x,t) = \frac{1}{2}(-h(0) + h(0)) + \frac{1}{2c}\int_{x-ct}^{x+ct} (-h'(0))_{\text{ODD}}\, dy + \iint_\Delta (f - h'')_{\text{ODD}}.$$

Notice that since the initial condition $v(x,0) = -h(0)$ is a constant, we used the odd extension $+h(0)$ at the left endpoint $x - ct$, which is less than 0 in this region. Similarly, for the extension of the initial velocity, we use $+h'(0)$ to the left of 0 and $-h'(0)$ to the right. Therefore,

$$\begin{aligned} v(x,t) &= \frac{1}{2c}\int_{x-ct}^{0} h'(0)\, dy - \frac{1}{2c}\int_0^{x+ct} h'(0)\, dy - \frac{1}{2c}\iint_\Delta h''_{\text{ODD}} + \frac{1}{2c}\iint_\Delta f_{\text{ODD}} \\ &= \frac{h'(0)}{2c}(-x+ct) - \frac{h'(0)}{2c}(x+ct) - \frac{1}{2c}\iint_\Delta h''_{\text{ODD}} + \frac{1}{2c}\iint_\Delta f_{\text{ODD}} \\ &= \frac{x}{c}h'(0) - \frac{1}{2c}\iint_\Delta h''_{\text{ODD}} + \frac{1}{2c}\iint_\Delta f_{\text{ODD}}. \end{aligned} \tag{3}$$

Now we only need to carefully compute the double integral of $h''_{\text{ODD}}$ on $\Delta$. One way to do this is to break up the region $\Delta$ into four parts (see the picture below). Region I is the only one where the odd extension is used: on this region $h''_{\text{ODD}}(y,s) = -h''(s)$. (Everywhere else on $\Delta$, $h''_{\text{ODD}}(y,s) = h''(s)$.)




Since for each fixed $s$, the function $h''(s)$ is a constant, by symmetry we have that

$$-\iint_{I\cup II} h''_{\text{ODD}}(y,s)\,dy\,ds = \iint_I h''(s)\,dy - \iint_{II} h''(s) = 0.$$

Then, on the square region:

$$-\iint_{III} h''(s)\,ds = -\frac{1}{2c}\int_0^{t-\frac{x}{c}}\int_0^{2x} h''(s)\,ds\,dy = -\frac{x}{c}\int_0^{t-\frac{x}{c}} h''(s)\,ds$$
$$= \frac{x}{c}h'(0) - \frac{x}{c}h'(t-\frac{x}{c}).$$

On the top triangle:

$$-\frac{1}{2c}\iint_{IV} h''(s)\,ds = -\frac{1}{2c}\int_{t-\frac{x}{c}}^{t}\int_{x-c(t-s)}^{x+c(t-s)} h''(s)\,dy\,ds.$$
$$= -\int_{t-\frac{x}{c}}^{t}(t-s)h''(s)\,ds = \int_{t-\frac{x}{c}}^{t}(-1)h'(s)\,ds - (t-s)h'(s)\Big|_{s=(t-x/c)}^{t}$$
$$= h(t-\frac{x}{c}) - h(t) + \frac{x}{c}h'(t-\frac{x}{c}).$$

Adding the integrals over each of these four regions, we find

$$-\frac{1}{2c}\iint_\Delta h''_{\text{ODD}} = h(t-\frac{x}{c}) - h(t) + \frac{x}{c}h'(0).$$

*We could have alternatively used Green's theorem to find the same result:*

$$\begin{aligned}-\tfrac{1}{2c}\iint_\Delta h''_{ODD} &= \tfrac{1}{2c}\oint_C h'_{ODD}\,dy = \tfrac{1}{2c}\int_{x-ct}^{x-ct} h'_{ODD}(y,0)\,dy \\ &\quad + \tfrac{1}{2c}\int_{x+ct}^{x} h'(\tfrac{x}{c}+t-\tfrac{y}{c})\,dy + \tfrac{1}{2c}\int_{x}^{x-ct} h'_{ODD}(y,-\tfrac{x}{c}+t+\tfrac{y}{c})\,dy \\ &= \tfrac{x}{c}h'(0) - h(t) + h(t-\tfrac{x}{c}),\end{aligned}$$

*where $C$ is the curve bounding the domain of dependence. $C$ is made up of three lines: one is the line $s = 0$ between $x - ct$ and $x + ct$, and the other two are the characteristics of slope $\pm 1/c$ that pass through the point $(x, t)$ (notice that along these characteristics, $s = \pm x/c + t \mp y/c$).* Therefore, from (3), we have that

$$v(x,t) = \iint_\Delta f_{\text{ODD}} + h(t - \frac{x}{c}) - h(t).$$

In the region where $x < ct$, we have found that $u(x, t) = v(x, t) + h(t)$ is

$$u(x,t) = \iint_\Delta f_{\text{ODD}} + h(t - \frac{x}{c}).$$

This proves the second part of formula (2) for the solution of the wave equation on the half-line with Dirichlet boundary condition.

We could also directly check that formula (2) works! Show that it solves the wave equation with the source $f$ and also gives the right initial conditions and the right boundary condition at $t = 0$.