Math 124B: PDEs

Eigenvalue problems for differential operators

We want to find eigenfunctions of (linear) differential operators acting on functions on the interval $[0, l]$ that satisfy boundary conditions at the endpoints. (In this discussion, we will assume that the function 0 solves $A0 = 0$ and satisfies the boundary conditions.) For instance, we have often looked at the second-order differential operator $A = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ with two boundary conditions.

The eigenvalue problem for such an A (with boundary conditions) is to find all the possible *eigenvalues* of A. In other words, we have to find all of the numbers λ such that there is a solution of the equation $AX = \lambda X$ for some function X ($X \neq 0$) that satisfies the boundary conditions at 0 and at l. When λ is an eigenvalue, all of these non-zero solutions are *eigenfunctions* corresponding to λ.

If we have the right number of boundary conditions, we often find that only some special set of numbers will be eigenvalues. Imagine picking any number λ you want. You can always solve the ordinary differential equation $AX = \lambda X$. There will be actually be many solutions of the ODE! For example, if the ODE is second order, then the general solution will have two arbitrary constants A and B. We want to find out which of these solutions also satisfy the boundary conditions. If there are *two* boundary conditions, you will have two equations involving the constants A and B. Most of the time, there will be only one possible solution of these two equations with two unknowns – which means most of the time, 0 is the only function that solves the ODE and satisfies the boundary conditions! Therefore, most of the time, the λ you picked is *not* an eigenvalue. The number λ is an eigenvalue only if it happens to be a number that somehow allows your two equations to have more than one possible solution for A and B.

Let's see an example of this: Let A be the operator $-\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ that acts on functions on $[0, l]$ with boundary conditions $X(0) = 0$ and $X'(l) = 0$. We want to find all the λ such that

$$-\frac{\mathrm{d}^2}{\mathrm{d}x^2}(X) = \lambda X; \quad X(0) = 0; \quad X'(l) = 0 \tag{$\star$}$$

has a non-zero solution. When we write down the general solution, the two boundary conditions will give us equations for the arbitrary constants A and B and the number λ. Our goal is to find all the numbers λ such that when we solve these two equations for A and B, we do *not* get that the general solution must become $X = 0$. However, for this particular ODE, we can not write down the general solution without first knowing if λ is equal to zero, is positive, or is negative. Therefore, we consider each of these three cases separately.

- Case (i): $\lambda = 0$

  In this case, λ is a specific number, so we're really just checking whether or not 0 is an eigenvalue. The general solution is $X(x) = Ax + B$. The two boundary conditions give the equations

$$X(0) = B = 0$$
$$X'(l) = A = 0$$

Clearly, the only solution of these equations is $A = 0$ and $B = 0$. Therefore, the only solution of $(\star)$ is $X = 0$, which means 0 is not an eigenvalue.

- Case (ii): $\lambda < 0$

  When λ is a negative number, $\lambda = -\beta^2$ for some $\beta > 0$ and the general solution is $X(x) = A\text{cosh}(\beta x) + B\text{sinh}(\beta x)$. Using the boundary conditions, the two equations are

  $$X(0) = A = 0$$
  $$X'(l) = \beta A\text{sinh}(\beta l) + \beta B\text{cosh}(\beta l) = 0$$

  Since A must be 0, this system of two equations has a solution only when $\beta B\text{cosh}(\beta l) = 0$. Remember $\beta > 0$ and $\cosh(a)$ never equals 0 for any number a. Therefore, B must be 0. The only solution is again $A = B = 0$, so $X(x) = 0$, and λ cannot be an eigenvalue. (I.e., there can be no negative eigenvalues.)

- Case (iii): $\lambda > 0$

  In this case, $\lambda = \beta^2$ for some $\beta > 0$ and the general solution is $X(x) = A\cos(\beta x) + B\sin(\beta x)$. The two boundary conditions give us the following system of equations:

  $$X(0) = A = 0$$
  $$X'(l) = -\beta A\sin(\beta l) + \beta B\cos(\beta l) = 0.$$

  Since $A = 0$, this system is solved only when $A = 0$ and $\beta B\cos(\beta l) = 0$. For *most* β, this means $B = 0$, so $X = 0$ and β^2 is not an eigenvalue. However, when β is $\frac{\pi}{2l}, \frac{3\pi}{2l}, \frac{5\pi}{2l}$, etc, then $\cos(\beta l) = 0$, and B does not have to be 0! These means that if $\lambda = \lambda_n = \left(\frac{(2n+1)\pi}{2l}\right)^2$ for some $n = 0, 1, 2, ...$ then λ is an eigenvalue, and the eigenfunction is $X_n(x) = B\sin\left(\frac{(2n+1)\pi}{2l}x\right)$ (A must still be 0, but B could be anything and X_n will still satisfy the boundary conditions!). If λ is any other positive number, it's not an eigenvalue.

We've solved the eigenvalue problem: The only eigenvalues are the λ_n, for $n = 0, 1, 2, ...$! (In the process of figuring out which numbers were eigenvalues, notice that we had to solve for the eigenfunctions X_n as well! This is because the definition of eigenvalue involves the existence of an eigenfunction.)