Math 124B: PDEs

Example from class (§5.6)

Solve the following wave equation on the finite interval $(0,\pi)$ *with a source:*

$$u_{tt} - u_{xx} = x^2$$

with boundary conditions $u_x(0,t) = 1$ *and* $u_x(\pi,t) = 0$ *and with initial conditions* $u(x,0) = 2\cos(5x)$ *and* $u_t(x,0) = 0$.

We need to use the eigenfunctions $X_o(x) = 1$ and $X_n(x) = \cos(nx)$ $(n = 1,2,3...)$ since we have Neumann boundary conditions at both endpoints. We want to solve for $u(x,t)$, and since the PDE includes $u_{tt}(x,t)$, $u_{xx}(x,t)$, and x^2, we will expand all of these functions as Fourier cosine series:

$$u(x,t) = \frac{u_o(t)}{2} + \sum_{n=1}^{\infty} u_n(t)\cos(nx)$$

$$u_{tt}(x,t) = \frac{v_o(t)}{2} + \sum_{n=1}^{\infty} v_n(t)\cos(nx)$$

$$u_{xx}(x,t) = \frac{w_o(t)}{2} + \sum_{n=1}^{\infty} w_n(t)\cos(nx)$$

$$x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}(-1)^n\frac{4}{n^2}\cos(nx).$$

We know the cosine series for x^2 from previous homework. We do not know yet what the coefficients u_n, v_n, and w_n are — notice that $u_n(t)$ is what we are trying to solve for! But we do know that v_n and w_n are related to u_n: Using the formulas for calculating Fourier coefficients, we have the following formulas for v_n and w_n:

$$\begin{aligned}
v_n(t) &= \frac{2}{\pi}\int_0^\pi u_{tt}(x,t)\cos(nx)\,dx = \frac{2}{\pi}\int_0^\pi \frac{\partial^2}{\partial t^2}\left[u(x,t)\cos(nx)\right]\,dx \\
&= \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left[\frac{2}{\pi}\int_0^\pi u(x,t)\cos(nx)\right]\,dx = \frac{\mathrm{d}^2 u_n}{\mathrm{d}t^2}(t) \qquad (1)\\
w_n(t) &= \frac{2}{\pi}\int_0^\pi u_{xx}(x,t)\cos(nx)\,dx = n\frac{2}{\pi}\int_0^\pi u_x(x,t)\sin(nx)\,dx + \frac{2}{\pi}u_x(x,t)\cos(nx)\ \Big|_{x=0}^{\pi} \\
&= -n^2\frac{2}{\pi}\int_0^\pi u_x(x,t)\cos(nx)\,dx + \frac{2n}{\pi}u(x,t)\sin(nx)\ \Big|_{x=0}^{\pi} - \frac{2}{\pi} \qquad (2)\\
&= -n^2\frac{2}{\pi}\int_0^\pi u_x(x,t)\cos(nx)\,dx - \frac{2}{\pi} = -n^2 u_n(t) - \frac{2}{\pi}.
\end{aligned}$$

Notice that these formulas work for all $n = 1, 2, 3, ..$, and also for $n = 0$! Going back to the PDE $u_{tt} - u_{xx} = x^2$, we can plug in the Fourier series for u_{tt}, u_{xx}, and x^2 to find

$$\frac{v_o(t)}{2} + \sum_{n=1}^{\infty} v_n(t)\cos(nx) - \frac{w_o(t)}{2} - \sum_{n=1}^{\infty} w_n(t)\cos(nx) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}(-1)^n \frac{4}{n^2}\cos(nx)$$

$$\frac{v_o(t) - w_o(t)}{2} + \sum_{n=1}^{\infty}(v_n(t) - w_n(t))\cos(nx) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}(-1)^n \frac{4}{n^2}\cos(nx)$$

For the two cosine series above to be equal, we must have that the coefficients of each eigenfunction are the same — so the above equality gives us infinitely many equations:

$$v_o(t) - w_o(t) = \frac{2\pi^2}{3}$$

$$v_n(t) - w_n(t) = (-1)^n \frac{4}{n^2} \quad \text{for } n = 1, 2, 3, ...$$

Using $v_n(t) = u_n''(t)$ and $w_n(t) = -n^2 u_n(t) - \frac{2}{\pi}$ for all $n = 0, 1, 2, 3...$ (see (**??**) and (**??**)), these equations become the following ODEs for u_n:

$$\boxed{\begin{aligned} u_o''(t) &= -\frac{2}{\pi} + \frac{2\pi^2}{3} \\ u_n''(t) + n^2 u_n(t) &= -\frac{2}{\pi} + (-1)^n \frac{4}{n^2} \quad \text{for } n = 1, 2, 3, ... \end{aligned}}$$

For each n, to solve the second-order ODE, we need the two initial conditions $u_n(0)$ and $u_n'(0)$. We can find these using the initial conditions for the PDE. We have the Fourier series for $u(x, t)$ — plug in $t = 0$ and set this equal to the initial condition $u(x, 0) = 2\cos(5x)$:

$$\frac{u_o(0)}{2} + \sum_{n=0}^{\infty} u_n(0)\cos(nx) = 2\cos(5x).$$

Again, we need to match the constant terms on each side, so $u_o(0) = 0$, and we need to match the coefficients of $\cos(nx)$ for each n, so $u_n(0) = 0$ for $n = 1, 2, 3, 4, 6, 7, ...$, but $u_5(0) = 2$. For the initial velocity,

$$u_t(x, 0) = \frac{u_o'(0)}{2} + \sum_{n=0}^{\infty} u_n'(0)\cos(nx) = 0,$$

so clearly $u_n'(0) = 0$ for all n.

<u>For $n = 0$</u>, solve

$$u_o''(t) = -\frac{2}{\pi} + \frac{2\pi^2}{3}; \quad u_o(0) = 0; \quad u_o'(0) = 0.$$

The general solution of the ODE (integrate twice!) is $u_o(t) = \left(-\frac{1}{\pi} + \frac{\pi^2}{3}\right) t^2 + At + B$. Using the initial conditions $u_o(0) = B = 0$ and $u_o'(0) = A = 0$.

$$\boxed{u_o(t) = \left(-\frac{1}{\pi} + \frac{\pi^2}{3}\right) t^2}$$

For $n=5$, solve

$$u_5''(t) + 25\,u_5(t) = -\frac{2}{\pi} - \frac{4}{25}; \quad u_5(0) = 2; \quad u_5'(0) = 0.$$

The general solution of the ODE is $u_5(t) = A\cos(5t) + B\sin(5t) - \frac{2}{25\pi} - \frac{4}{625}$ (check by plugging it back into the ODE!) This is the sum of the general solution of the homogeneous equation $u_5'' + 25u_5 = 0$ and of one specific solution of the ODE. Using the boundary conditions, $u_5'(0) = 5B = 0$, so $B = 0$. Also, $u_5(0) = A - \frac{2}{25\pi} - \frac{4}{625} = 2$, so $A = 2 + \frac{2}{25\pi} + \frac{4}{625}$.

$$\boxed{u_5(t) = \left(2 + \frac{2}{25\pi} + \frac{4}{625}\right)\cos(5t) - \frac{2}{25\pi} - \frac{4}{625}}$$

For all other n $(n \neq 0, 5)$, solve

$$u_n''(t) + n^2 u_n(t) = -\frac{2}{\pi} + (-1)^n \frac{4}{n^2}; \quad u_n(0) = 0; \quad u_n'(0) = 0.$$

The general solution of the ODE is $u_n(t) = A\cos(5t) + B\sin(5t) - \frac{2}{n^2\pi} + (-1)^n \frac{4}{n^4}$. Using the initial conditions tells us $B = 0$ and $A = \frac{2}{n^2\pi} - (-1)^n \frac{4}{n^4}$.

$$\boxed{u_n(t) = \left(\frac{2}{n^2\pi} + (-1)^{n+1}\frac{4}{n^4}\right)(\cos(5t) - 1)}$$

Putting these solutions together, we know all of the Fourier coefficients $u_n(t)$ for the function $u(x,t)$, so we have solved for $u(x,t)$!

$$\boxed{\begin{aligned} u(x,t) = \tfrac{1}{2}\left(-\tfrac{1}{\pi} + \tfrac{\pi^2}{3}\right)t^2 + \left\{2\cos(5t) + \left(\tfrac{2}{25\pi} + \tfrac{4}{625}\right)(\cos(5t) - 1)\right\}\cos(5x) \\ + \sum_{\substack{n=1 \\ n\neq 5}}^{\infty} \left(\tfrac{2}{n^2\pi} + (-1)^{n+1}\tfrac{4}{n^4}\right)(\cos(5t) - 1)\cos(nx) \end{aligned}}$$