Math 124B: PDEs

Final Review

Review your homework problems, as well as the midterm and midterm review.

Outline

1. Eigenvalue problems

   - Know how to determine the *eigenvalues* along with the *eigenfunctions* of the operator $-\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ with given boundary conditions on an interval. (See examples in Chapter 4 and in the handout from class.)
   - What theorems do you know about the eigenvalues and eigenfunctions of this problem with *symmetric* boundary conditions? When are the eigenvalues also non-negative? (Section 5.3)

2. General Fourier series

   - Once you have a list of eigenvalues and corresponding eigenfunctions for the eigenvalue problem above, any function on the interval can be written as a *Fourier series* (an infinite linear combination of these eigenfunctions). How do you find the constants?
   - Notice that each different boundary condition will yield a different list of eigenfunctions — what are some of the common Fourier series that you know, and what boundary conditions do the eigenfunctions satisfy?
   - Understand the three theorems about the convergence of Fourier series.
   - What is Parseval's equality?

3. Separation of variables

   - Know how to find separated solutions of PDEs with given boundary conditions. For example, find all solutions of the form $u(x,t) = X(x)T(t)$ of the heat equation with the mixed boundary conditions $u'(0,t) = 0$ and $u(l,t) = 0$.
   - When the boundary conditions are *homogeneous*, the separated solutions can be summed up to give a general series solution to the problem. (Initial conditions would be needed to determine the constants in the series.)

4. Solving heat and wave equations using Fourier series

   - This works even for *inhomogeneous* boundary conditions! Of course, when the boundary conditions are homogeneous, this technique gives exactly the same solution as the general series solution above.

- Expand the solution $u(x,t)$ (and every other function that appears in the problem) as a Fourier series in x. Important: make sure you use the Fourier series with the eigenfunctions that correspond to the right boundary conditions! (See section 5.6 and the handout from class for examples.)

5. Fourier transform

   - What is the definition of the Fourier transform of a function $f(x)$? What does Fourier's identity say?
   - What is the definition of the convolution of two functions f and g? (This will be given on the exam.) Use the definition of Fourier transform and of convolution to show that the Fourier transform of the function $(f \star g)(x)$ is $F(k)G(k)$.
   - Know how to use the Fourier transform to solve PDEs (such as the heat equation, or the heat equation with a convection term — see §12.4, exercise #1).

6. Laplace's equation

   - Be able to find radial solutions to Laplace's equation. (The operator Δ in terms of polar and spherical coordinates will be given.)
   - Be able to solve $u_{xx} + u_{yy} = 0$ on a rectangle with given boundary conditions on each side. (Separate variables, so assume $u(x,y) = X(x)Y(y)$, and solve for X and Y using three homogeneous boundary conditions — after finding all of these separated solutions, combine them into a general series solution. Finally, use the (inhomogeneous) boundary condition on the fourth side of the rectangle to determine the constants in the series.)
   - To solve $u_{xx}+u_{yy} = 0$ on a circle (with any given boundary condition at $r = a$) use polar coordinates and separatation of variables to solve for $u(r,\theta) = R(r)\Theta(\theta)$ — what are the boundary conditions for Θ?. (From this, we find Poisson's formula for the solution of the Dirichlet problem on the circle — the formula will be given if needed.)

Problems

1. Consider the eigenvalue problem for $-\frac{d^2}{dx^2}$ with boundary conditions $X(0) = 0$ and $X'(l) + 2X(l) = 0$. Find equations that determine all of the eigenvalues, and list the corresponding eigenfunctions.

2. In what sense does the Fourier cosine series of $f(x) = 2\pi x - x^2$ converge on $(0, \pi)$? (I.e., does it converge in L^2, pointwise, or uniformly? Give reasons!) What does the series converge to on $(-\pi, 0)$?

3. (a) Find the complex Fourier series for x^2 on the interval $(0, \pi)$.

   (b) Use part (a) and Parseval's equality to find the value of the sum $\sum_{n=1}^{\infty} \frac{1}{n^4}$.

4. Solve for $u(x,t)$:

$$\begin{aligned} &u_{tt} = u_{xx} \qquad \text{for } 0 < x < 1 \\ &u(0,t) = 0; \quad u_x(1,t) = e^t \\ &u(x,0) = \sin(\tfrac{\pi}{2}x); \quad u_t(x,0) = 1 \end{aligned}$$

5. Solve Laplace's equation on the upper half-plane with the given boundary condition:

$$\begin{aligned} u_{xx} + u_{yy} &= 0 \quad \text{in the half-plane } y > 0 \\ u(x,0) &= \phi(x) \quad \text{on the line } y = 0. \end{aligned}$$

   Show all your steps! *Hint: Use the Fourier transform in the $x-$variable.*

6. (a) Solve $u_{xx}+u_{yy}+u_{zz} = 1$ in the spherical shell $a < r < b$ with the Dirichlet condition $u = 0$ on $r = a$ and the Neumann condition $\frac{\partial u}{\partial r} = 0$ on $r = b$.

   (b) What happens if you try to solve this problem on the sphere $r < b$, with the conditions $\frac{\partial u}{\partial r} = 0$ on $r = b$ and $u(0)$ is finite? Can you solve for u if you change the condition to $\frac{\partial u}{\partial r} = 1$?

7. Solve Laplace's equation on the quarter circle in the first quadrant ($0 < r < 1$ and $0 < \theta < \frac{\pi}{2}$) with the boundary conditions $u(r,0) = 0$, $u(r, \frac{\pi}{2}) = 0$, and $\frac{\partial u}{\partial r}(1,\theta) = \cos(\theta)$. (Also assume u is finite at $r = 0$.)

   (*Hint: Use $\Delta u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}$ and separate variables. What boundary conditions will you use to find the separated solutions? Once you have a general series solution, what boundary condition will you use to figure out the constants?*)