The Fourier sine series of the function $\phi(x) = \begin{cases} 0 & 0 < x < \frac{1}{4} \\ 1 & \frac{1}{4} < x < \frac{3}{4} \\ 0 & \frac{3}{4} < x < 1 \end{cases}$ is

$$\sum_{n=1}^{\infty} \frac{2}{n\pi} \left[\cos(\frac{1}{4}n\pi) - \cos(\frac{3}{4}n\pi)\right] \sin(n\pi x)$$

By the convergence theorem for classical Fourier series, this series converges for all x! What happens at $x = \frac{1}{4}$? What happens very nearby $x = \frac{1}{4}$?

The sum of first N terms in the series for the given value of x (to four decimal places)

	x = 0.25	x = 0.251	x = 0.249
N=1	0.6366	0.6386	0.6346
N=2	0.4244	0.4284	0.4204
N=3	0.5517	0.5577	0.5457
N=4	0.4608	0.4688	0.4528
N=5	0.5315	0.5415	0.5215
N=10	0.4841	0.5042	0.4642
N=50	0.4968	0.5964	0.3975
N=500	0.4977	1.0898	-0.0892
N=5000	0.5000	0.9899	0.0101
N=50000	0.5000	0.9990	0.0010

In general, for points very close to the jump, the convergence of the series will take a long time. For example, look at the partial sums of the series for $x = 0.250001$: when $N = 500$, the sum is only 0.5007; when $N = 50,000$, it is 0.5995; when $N = 100,000$, it is 0.6957; when $N = 200,000$, it is 0.8665; when $N = 300,000$, it is 0.9935; when $N = 500,000$, it is 1.0895; and when $N = 5,000,000$, it is 0.9899. So the series eventually converges to 1 – although even for $N = 50,000,000$, the partial sum is still only 0.9990!