Math 124B: PDEs

Solving the heat equation with the Fourier transform

Find the solution $u(x,t)$ of the diffusion (heat) equation on $(-\infty,\infty)$ with initial data $u(x,0)=\phi(x)$.

We will need the following facts (which we prove using the definition of the Fourier transform):

- $\widehat{\mathbf{u_t}}(\mathbf{k},\mathbf{t}) = \dfrac{\partial}{\partial \mathbf{t}}\widehat{\mathbf{u}}(\mathbf{k},\mathbf{t})$ Pulling out the time derivative from the integral:

$$\begin{aligned}\widehat{u_t}(k,t) &= \int_{-\infty}^{\infty} u_t(x,t)e^{-ikx}\,dx = \int_{-\infty}^{\infty} \frac{\partial}{\partial t}\left[u(x,t)e^{-ikx}\right]\,dx = \frac{\partial}{\partial t}\left[\int_{-\infty}^{\infty} u(x,t)e^{-ikx}\,dx\right] \\ &= \frac{\partial}{\partial t}[\widehat{u}(k,t)].\end{aligned}$$

- $\widehat{\mathbf{u_{xx}}}(\mathbf{k},\mathbf{t}) = (\mathbf{ik})^{\mathbf{2}}\,\widehat{\mathbf{u}}(\mathbf{k},\mathbf{t}))$ Integrating by parts twice:

$$\begin{aligned}\widehat{u_{xx}}(k,t) &= \int_{-\infty}^{\infty} u_{xx}(x,t)e^{-ikx}\,dx = -\int_{-\infty}^{\infty} u_x(x,t)[(-ik)e^{-ikx}]\,dx \\ &= (ik)\int_{-\infty}^{\infty} u_x(x,t)e^{-ikx}\,dx = (ik)^2\int_{-\infty}^{\infty} u(x,t)e^{-ikx}\,dx \\ &= (ik)^2\,\widehat{u}(k,t).\end{aligned}$$

We know that $u_t - \alpha u_{xx} = 0$ (for some constant $\alpha > 0$) and $u(x,0)=\phi(x)$. Taking the Fourier transform of both of these equations tells us that

$$\frac{\partial}{\partial t}\widehat{u}(k,t) + \alpha k^2\,\widehat{u}(k,t) = 0 \quad \text{and} \quad \widehat{u}(k,0) = \widehat{\phi}(k).$$

Mulitiplying both sides of the first equation by the integrating factor $e^{\alpha k^2 t}$, the equation becomes

$$\frac{\partial}{\partial t}\left(e^{\alpha k^2 t}\widehat{u}(k,t)\right) = 0.$$

When we integrate *with respect to* t (so hold k fixed!), this becomes $e^{\alpha k^2 t}\widehat{u}(k,t) = f(k)$ where $f(k)$ is an arbitrary function of k. Then,

$$\widehat{u}(k,t) = f(k)e^{-\alpha k^2 t}.$$

Using the initial condition $\widehat{u}(k,0) = \widehat{\phi}(k)$, we find out that $f(k) = \widehat{\phi}(k)$. (*Notice that if we forgot that when we integrate with respect to t, the arbitrary constant is really a function of k, then we wouldn't be able to satisfy the initial condition.*) Now we know $\widehat{u}(k,t) = \widehat{\phi}(k)e^{-\alpha k^2 t}$, but what we want to know is the solution $u(x,t)$ in terms of the original variable x. What we are really doing is looking for the function $u(x,t)$ whose Fourier transform is $\widehat{\phi}(k)e^{-\alpha k^2 t}$! The first step is just to find the function $S(x,t)$ whose Fourier transform is $\widehat{S}(k,t) = e^{-\alpha k^2 t}$. Using Fourier's identity,

$$S(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{S}(k,t)e^{ikx}\,dk = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\alpha k^2 t + ikx}\,dk = \frac{1}{\sqrt{4\pi\alpha t}}\,e^{-\frac{1}{4\alpha t}x^2}.$$

(For the last step, we can compute the integral by completing the square in the exponent. Alternatively, we could have just noticed that we've already computed that the Fourier transform of the Gaussian function $\frac{1}{\sqrt{4\pi\alpha t}}\,e^{-\frac{1}{4\alpha t}x^2}$ *gives us* $e^{-\alpha k^2 t}$*.)*

Finally, we need to know the fact that Fourier transforms turn convolutions into multiplication. Therefore, to get the Fourier transform $\widehat{u}(k,t) = e^{-\alpha k^2 t}\widehat{\phi}(k) = \widehat{S}(k,t)\widehat{\phi}(k)$, we must have started with the function $u = S \star \phi$: From the definition of the convolution,

$$u(x,t) = (S(.,t) \star \phi(.))(x) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)\,dy = \boxed{\frac{1}{\sqrt{4\pi\alpha t}}\int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4\alpha t}}\phi(y)\,dy}\,.$$

This is the solution of the heat equation for any initial data ϕ. We derived the same formula last quarter, but notice that this is a much quicker way to find it!