Math 124B: PDEs

Midterm Review

1. Eigenvalue problems
   (a) Know how to solve an eigenvalue problem for an operator on a finite interval with given boundary conditions. (See handout!)
   (b) Examples from homework: Section 4.3, #9, #16; Section 5.3, #6
2. Separation of Variables
   (a) Know how to find the *separated solutions* of a PDE with given boundary conditions. For all of the second order equations we've looked at, you need to write $u(x,t) = X(x)T(t)$ and find the system of ODEs that X and T satisfy, along with boundary conditions for X. Solve the eigenvalue problem for X, and for each possible solution X (i.e., for each eigenfunction!), find the corresponding solution for T. This will give you all of the separated solutions.
   (b) Recall that the sum (i.e., linear combination) of the separated solutions is the *general series solution* for the PDE.
   (c) Example: Find the general series solution for the diffusion (heat) equation with Neumann boundary conditions on $[0, 1]$.
   (d) Examples from homework: Section 4.1, #3; Section 4.2, #2
3. Fourier Series
   (a) If a function $\phi(x)$ on $(0, l)$ is given by a Fourier sine series or a Fourier cosine series, know how to find the coefficients.
   (b) If we have a general series solution of a PDE with given boundary conditions, we can use Fourier series of the initial conditions to find the constants in the general solution.
   (c) Examples from homework: Section 5.1, #2; Section 5.1, #8; Section 5.3, #3
   (d) Understand the full Fourier series (real or complex) on $(-l, l)$, and properties of even, odd, and periodic functions.
4. Orthogonality
   (a) Know the definition of the *inner product* (f, g) of two functions $f(x)$ and $g(x)$ on the interval $[a, b]$. (Remember the functions might be complex-valued!) What does it mean for $f(x)$ and $g(x)$ to be orthogonal? Also, define the size of a function $f(x)$ on $[a, b]$ to be $\|f\|^2 = (f, f)$. (Notice that $\|f\| > 0$ if $f(x) \not\equiv 0$.)
   (b) Examples: Section 5.3, # 2; Show that $\sin(\frac{n\pi}{l}x)$ and $\cos(\frac{m\pi}{l}x)$ are orthogonal on $(-l, l)$ for all $n = 1, 2, 3, ...$ and $m = 1, 2, 3, ...$
   (c) We have three theorems involving general Fourier series and orthogonality. The following definition will be given on the exam: The boundary conditions

$$\alpha_1 X(a) + \beta_1 X(b) + \gamma_1 X'(a) + \delta_1 X'(b) = 0$$
$$\alpha_2 X(a) + \beta_2 X(b) + \gamma_2 X'(a) + \delta_2 X'(b) = 0$$

are *symmetric* if

$$\left[-f(x)\overline{g'(x)} + f'(x)\overline{g(x)}\right] \Big|_{x=a}^{b} = 0$$

You should know that the following theorems are true and also know how to prove them by integration by parts.

i. Symmetric BCs $\Rightarrow$ all eigenvalues are real
ii. Symmetric BCs $\Rightarrow$ if X_1 and X_2 are two functions corresponding to two different eigenvalues $\lambda_1 \neq \lambda_2$, then X_1 and X_2 are orthogonal
iii. If the BCs are symmetric and if all real-valued functions $f(x)$ that satisfy the BCs also satisfy $f(x)f'(x)\big|_{x=a}^{b} \leq 0$, then there are no negative eigenvalues.

5. Convergence of Fourier Series

(a) Three different types of convergence for series

i. Know the definitions for L^2 *convergence*, *pointwise convergence*, and *uniform convergence* of infinite series.
ii. Example: Notice that (for $-1 < x < 1$) $\sum_{n=0}^{2N}(-1)^n x^{2n} = \sum_{n=0}^{2N}(-x^2)^n = \frac{1+x^{4N+2}}{1+x^2}$. Explain why $\sum_{n=0}^{\infty}(-1)^n x^{2n}$ converges pointwise, but not uniformly in the interval $-1 < x < 1$. What is the pointwise limit? Does it converge in the L^2 sense?
iii. Example from homework: Section 5.4, # 3

(b) Three convergence theorems

i. For all of the below theorems, we assume we have solved eigenvalue problem $-X'' = \lambda X$ on (a,b) with symmetric BCs. Therefore, we have infinitely many eigenvalues (increasing to $+\infty$) along with corresponding orthogonal eigenfunctions $\{X_n\}_{n=1}^{\infty}$. How do you define the *Fourier coefficients* and the *Fourier series* of any function $f(x)$ on (a,b)?
ii. L^2 **Convergence Theorem.** *If $\|f(x)\|$ is finite, its Fourier series converges in the L^2 sense.*
iii. **Pointwise Convergence Theorem.** *If $f(x)$ and $f'(x)$ are piecewise continuous on $a \leq x \leq b$, the* classical *Fourier series for $f(x)$ converges almost everywhere.* Know what the Fourier series converges to at points where f is not continuous!
iv. **Uniform Convergence Theomem.** *If $f(x)$, $f'(x)$, and $f''(x)$ exist and are continuous for $a \leq x \leq b$ and $f(x)$ satisfies the boundary conditions, then the Fourier series converges* uniformly.
v. You will be given the three theorems above on the exam, but you will need to know what the assumptions mean and the definitions of the types of convergence!
vi. Example: In what sense does the Fourier sine series for $f(x) = 1$ on $(0,l)$ converge? What about the Fourier cosine series?
vii. Example: Let $\phi(x) = 0$ for $0 \leq x < 1$; $\phi(1) = -1$; and $\phi(x) = 1$ for $1 < x \leq 3$. Does the Fourier cosine series converge in L^2? uniformly? pointwise? What does it converge to when $x = 1$?
viii. Example from homework: Section 5.4, #8