Continuum Mechanics: Fall 2007

Review Problems

These problems are due anytime before December 17th. Please work on your own, using your book and notes. You can email me for office hours by appointment during finals week.

Problem #1
(a) Let $\mathbf{G}(\mathbf{A}) = \text{tr}\,(\mathbf{A}^2)\mathbf{A}^T$. At a point $\mathbf{A} \in \text{Lin}$, what is the domain and range of the derivative map $D\mathbf{G}(\mathbf{A})$? Find the derivative map $D\mathbf{G}$.

(b) (**§3 #7**) Let $\mathbf{G} : \text{Lin} \to \text{Lin}$ be differentible and satisfy $\mathbf{QG}(\mathbf{A})\mathbf{Q}^T = \mathbf{G}(\mathbf{QA})$ for all $\mathbf{A} \in \text{Lin}$ and $\mathbf{Q} \in \text{Orth}$. Show that

$$\mathbf{G}(\mathbf{A})\mathbf{W}^T + \mathbf{WG}(\mathbf{A}) = D\mathbf{G}(\mathbf{A})[\mathbf{WA}].$$

See the hint in the book (and the appendix §36).

Problem #2 (§13 #5) Prove that

$$\mathscr{K} = \mathscr{K}_\alpha + \frac{1}{2}m(\mathscr{B})\dot{\boldsymbol{\alpha}}^2$$

where $\mathscr{K}$ is the kinetic energy and $\mathscr{K}_\alpha$ is the relative kinetic energy (defined in terms of the velocity relative to the center of mass: $\mathbf{v}_\alpha(\mathbf{x}, t) = \mathbf{v}(\mathbf{x}, t) - \dot{\boldsymbol{\alpha}}(t)$.

$$\mathscr{K}(t) = \frac{1}{2}\int_{\mathscr{B}_t} \mathbf{v}^2(\mathbf{x}, t)\rho(\mathbf{x}, t)\, dV_x; \qquad \mathscr{K}_\alpha = \frac{1}{2}\int_{\mathscr{B}_t} \mathbf{v}_\alpha^2(\mathbf{x}, t)\rho(\mathbf{x}, t)\, dV.$$

Problem #3 Consider the motion (defined for $t < 2$) given in cartesian coordinates:

$$\begin{aligned} x_1 &= p_1 \\ x_2 &= p_2 + \frac{t}{2}p_3 \\ x_3 &= p_3 + \frac{t}{2}p_2. \end{aligned}$$

(a) Find the deformation gradient of the motion, and find the spatial velocity field $\mathbf{v}$ and its divergence and curl. Decide whether or not this motion is homogeneous, isochoric, irrotational, or steady.

(b) What are the pathlines of this motion? At $t = 1$, what has happened to the cube in the reference configuration with corners $(\pm 1, \pm 1, \pm 1)$? Write down the differential equations that the streamlines satisfy.

(c) Find the velocity gradient $\mathbf{L}$. Verify that $\dot{\mathbf{F}} = \mathbf{L}_m\mathbf{F}$. Find the spin tensor $\mathbf{W}$ and its axial vector.

Problem #4 A Cauchy stress tensor $\mathbf{T}$ is **pure shear** if there exists an orthonormal basis such that the diagonal components of $\mathbf{T}$ are 0.

(a) Prove that a stress tensor $\mathbf{T}$ is pure shear if and only if $\operatorname{tr}\mathbf{T} = 0$.

(b) Given any Cauchy stress tensor $\mathbf{T}$, prove that $\mathbf{T}$ can be uniquely decomposed as $\mathbf{T} = -p\mathbf{I} + \tilde{\mathbf{T}}$ for some constant p and for some symmetric tensor $\tilde{\mathbf{T}}$ such that $\operatorname{tr}\tilde{\mathbf{T}} = 0$ (i.e., for some pure shear $\tilde{\mathbf{T}}$).

(c) Consider a stress tensor $\mathbf{T} = (-2, -5, -10)$ In the standard cartesian coordinates, $\mathbf{T}$ is given by the matrix

$$[\mathbf{T}] = \begin{bmatrix} \text{-3} & \sqrt{3} & 0 \\ \sqrt{3} & \text{-5} & 0 \\ 0 & 0 & \text{-10} \end{bmatrix}$$

Find the planes on which there is no shear (that is, find the principle directions). What are the principle stresses? Also find the normal vector of the plane on which the shear stress is as large as possible.

Problem #5
Consider a motion χ of a body $\mathscr{B}$. For any part $\mathscr{P}$, the **flux** of $\mathbf{v}$ through $\partial\mathscr{P}_t$ is defined by

$$\Phi(\mathscr{P}, t) = \int_{\partial\mathscr{P}_t} \mathbf{v}\cdot\mathbf{m}\,dA.$$

You may use the fact that for the surface integral of a spatial vector field $\mathbf{g}$, the change of variables formula is given by

$$\int_{f(S)} \mathbf{g}(x)\cdot\mathbf{m}(x)\,dA_x = \int_S \mathbf{g}(f(p))\cdot\mathbf{G}(p)\mathbf{n}(p)\,dA_p.$$

where f is a deformation of the surface S with deformation gradient F, and where $\mathbf{G}(p) = [\det\mathbf{F}(p)]\mathbf{F}^{-T}(p)$. (See §6, $(14)_2$.)

(a) Find the material derivative of the flux $\dot{\Phi}(\mathscr{P}, t) = \frac{\mathrm{d}}{\mathrm{dt}}\Phi(\mathscr{P}, t)$. (Use the change of variables above and not the divergence theorem!)

(b) Use the divergence theorm and Reynold's transport theorem to find

$$\dot{\Phi}(\mathscr{P}, t) = \frac{\mathrm{d}}{\mathrm{dt}}\int_{\mathscr{P}_t} \operatorname{div}\mathbf{v}\,dV.$$

Show this is the same result as in (a) (you'll need to use the divergence theorem again).