The Inverse Function Theorem: Let $f : S \to \mathbb{R}^n$, $S \subseteq \mathbb{R}^n$, (where $\mathbb{R}^n$ is $n$-dimensional Euclidean space) be a smooth function. We will write $f = (f_1, ..., f_n)$, where each function $f_j : S \to \mathbb{R}$. Also, let $T = f(S)$. Assume that the Jacobian $J[f](x_0) \neq 0$ at a point $x_0 \in S$. Then, there is a unique function $g : Y \to X$, for some open sets $x_0 \in X \subseteq S$, $f(x_0) \in Y \subseteq T$, such that $f$ is one-to-one on $X$ and $f(X) = Y$, $g$ is smooth and has the property that $g(f(x)) = x$ for every $x \in X$.

We often think of the map $f$ as defining a change of coordinates (from the $(x_1, ..., x_n)$ variables to the $(f_1, ..., f_n)$ variables). Then, a non-zero Jacobian at a point $x_0$ implies that this change of coordinates is invertible (at least in a neighborhood of $x_0$), and that the inverse map is smooth. In other words, if you know a point in terms of the $f_j$ variables, you should be able to describe the point in terms of the $x_j$ variables.

For example, consider the map from $x = (x, y) \in \mathbb{R}^2 \setminus \{(0,0)\}$ to $(r, \theta) \in \mathbb{R}^2$ given by the functions $r = f_1(x, y) = \sqrt{x^2 + y^2}$ and $\theta = f_2(x, y) = \arctan \left( \frac{y}{x} \right)$. We can compute the Jacobian matrix of this mapping

$$
\begin{pmatrix}
\frac{x}{\sqrt{x^2 + y^2}} & \frac{y}{\sqrt{x^2 + y^2}} \\
\frac{y}{x} & \frac{x}{x^2 + y^2}
\end{pmatrix}
$$

The determinant of this matrix is $J[f](x) = (\sqrt{x^2 + y^2})^{-1}$. If $x_0$ is any non-zero point, there must exist a smooth inverse map (defined in some neighborhood of $(\sqrt{x_0^2 + y_0^2}, \arctan \left( \frac{y_0}{x_0} \right))$ that describes points $(x, y)$ in terms of the $(r, \theta)$ variables.

Another example is $f(x, y) = (e^x \cos(y), e^x \sin(y))$. The Jacobian of this function is $J[f](x, y) = e^{2x}$. Therefore, for every point $(x_0, y_0)$, there is a neighborhood of the point $(e^{x_0} \cos y_0, e^{x_0} \sin y_0)$ on which $f$ is invertible. Notice that even though $f$ is one-to-one on some neighborhood of every single point, $f$ is not one-to-one on the entire set $\mathbb{R}^2$, and so is not globally invertible!

For the proof, see, for example, Mathematical Analysis: A Modern Approach to Advanced Calculus by Tom M. Apostol or Principles of Mathematical Analysis by Walter Rudin.