Tensor Algebra I

Continuum Mechanics: Fall 2007

Reading: Gurtin, Section 1

Notation. Denote the space of 3-dimensional Euclidean points by $\mathscr{E}$, and the associated vector space by $\mathscr{V}$.

Representation Theorem for Linear Forms. Let $\psi : \mathscr{V} \to \mathbb{R}$ be linear. Then there exists a unique vector $\mathbf{a}$ such that $\psi(\mathbf{v}) = \mathbf{a} \cdot \mathbf{v}$ for every $\mathbf{v} \in \mathscr{V}$.

Note. Similar statements are true for many abstract inner product spaces!

Definition. A **tensor** is a linear transformation from $\mathscr{V}$ into $\mathscr{V}$.

Note. In general a *tensor* is a *multilinear form* from a vector space into $\mathbb{R}$, but our book uses tensor, as above, to always mean a "tensor of type $\binom{1}{1}$," or in other words, a *bilinear form* from $\mathscr{V} \times \mathscr{V}^*$ into $\mathbb{R}$.

The space of all tensors is a *vector space* with addition and scalar multiplication properly defined; also, the product of any two tensors is defined by composition. This space is also an *inner product space*, with the inner product defined by $\mathbf{S}\cdot\mathbf{T} = \operatorname{tr}(\mathbf{S}^T\mathbf{T})$, where the **transpose** of a tensor $\mathbf{S}$ is the unique tensor such that

$$\mathbf{S}\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{S}^T\mathbf{v} \text{ for every } \mathbf{u}, \mathbf{v} \in \mathscr{V},$$

and the **trace** is defined as the unique linear operator tr from the space of all tensors into $\mathbb{R}$ such that

$$\operatorname{tr}(\mathbf{u} \otimes \mathbf{v}) = \mathbf{u} \cdot \mathbf{v} \text{ for every } \mathbf{u}, \mathbf{v} \in \mathscr{V}.$$

Definition. The **tensor product** of any two vectors $\mathbf{a}$ and $\mathbf{b}$ is the tensor $\mathbf{a} \otimes \mathbf{b}$ defined by

$$\mathbf{a} \otimes \mathbf{b} = (\mathbf{b} \cdot \mathbf{v})\mathbf{a} \text{ for every } \mathbf{v} \in \mathscr{V}.$$

Recall from geometry that for any unit vector $\mathbf{e}$, $(\mathbf{e} \otimes \mathbf{e})\mathbf{v} = (\mathbf{v} \cdot \mathbf{e})\mathbf{e}$ is the projection of $\mathbf{v}$ in the direction $\mathbf{e}$; also $(\mathbf{I} - \mathbf{e} \otimes \mathbf{e}) = \mathbf{v} - (\mathbf{v} \cdot \mathbf{e})\mathbf{e}$ is the projection into the plane perpendicular to $\mathbf{e}$.

Definition. Given a skew tensor $\mathbf{W}$, there exists a unique vector $\mathbf{w}$, called the **axial vector** corresponding to $\mathbf{W}$, such that $\mathbf{W}\mathbf{v} = \mathbf{w} \times \mathbf{v}$ for all $\mathbf{v} \in \mathscr{V}$.

Note. This says that (after fixing a basis) a skew tensor $\mathbf{W}$ is determined by only three numbers (the components of its axial vector!).

Recall that the area of a rhombus with sides $\mathbf{w}$ and $\mathbf{v}$ (and angle θ between them) is $|\mathbf{v}\times\mathbf{w}| = |\mathbf{v}||\mathbf{w}|\sin\theta$. The volume of a parallelepiped with sides $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ is the absolute value of the scalar triple product

$$\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = \mathbf{w}\cdot(\mathbf{u}\times\mathbf{v}) = \mathbf{v}\cdot(\mathbf{w}\times\mathbf{u}).$$

Also, recall the relationship between the cross product and the determinant, and that the determinant measures the ratio of volumes: For any parallelepiped $\mathscr{P}$ with sides $\mathbf{u}, \mathbf{v}, \mathbf{w}$,

$$\det\mathbf{S} = \frac{\mathbf{Su}\cdot(\mathbf{Sv}\times\mathbf{Sw})}{\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})} = \pm\frac{\mathrm{vol}(\mathbf{S}(\mathscr{P}))}{\mathrm{vol}(\mathscr{P})}$$

In other words, if a volume is transformed by a linear map $\mathbf{S}$, a unit of volume is transformed into a volume of size $|\det\mathbf{S}|$.

Notation and definitions. Some special subsets of tensors are Sym, all *symmetric* tensors; Skw, all *skew* tensors; Psym, all positive definite, symmetric tensors; Orth, all orthogonal tensors; Orth^+, all rotations.

A tensor $\mathbf{Q}$ is **orthogonal** if $\mathbf{Qu} = \mathbf{Qv}$ for all $\mathbf{u}, \mathbf{v} \in \mathscr{V}$. Note that this is equivalent to $\mathbf{Q}^T\mathbf{Q} = \mathbf{I}$ or to $\mathbf{Q}\mathbf{Q}^T = \mathbf{I}$ or $\det\mathbf{Q} = \pm 1$. An orthogonal tensor $\mathbf{Q}$ is a **rotation** if its determinant is positive (and therefore, if and only if $\det\mathbf{Q} = +1$).

A tensor $\mathbf{S}$ is **positive definite** if $\mathbf{v}\cdot\mathbf{Sv} > 0$ for all $\mathbf{v} \in \mathscr{V}\setminus\{\mathbf{0}\}$.

Exercises. 3, 5, 4, 6, 7 (only (4) and (5)), 9, 11, 12, 15, 16, 17

- Fix a basis $\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}$ of $\mathbb{R}^3$. Consider the tensor (linear mapping) from $\mathbb{R}^3$ to $\mathbb{R}^3$ given by

$$\mathbf{S} = \begin{pmatrix} -1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and the vectors

$$\mathbf{v} = \begin{pmatrix} 4 \\ 3 \\ 0 \end{pmatrix}; \quad \mathbf{w} = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$$

Graph the vectors $\mathbf{v}$ and $\mathbf{w}$ (starting at the origin) on the $\mathbf{e_1} - \mathbf{e_2}$ plane. Sketch the rhombus formed by these two vectors and find its area (using geometry – recall the formula for the area of a rhombus and that the angle between any two non-zero vectors $\mathbf{v}$ and $\mathbf{w}$ is defined by $\cos^{-1}\frac{\mathbf{v}\cdot\mathbf{w}}{|\mathbf{v}||\mathbf{w}|}$). Then, calculate $|\mathbf{v}\times\mathbf{w}|$. Next, sketch the vectors $\mathbf{Sv}$ and $\mathbf{Sw}$ and the rhombus they form on another plane. Find the area of this rhombus and calculate $|\mathbf{Sv}\times\mathbf{Sw}|$. What must $\det\mathbf{S}$ be?

Notice that $\mathbf{S}$ is a 1-1 mapping from every point in the first rhombus to every point in the second rhombus (where we identify points $\mathbf{p}$ with vectors $\mathbf{p} - \mathbf{o}$). What happens if we transform the first rhombus using the mapping

$$T = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix}?$$