Math 5B: Critical points and absolute extreme values

Examples

Solving for Critical Points

Finding critical points is really an algebra problem: Solve the system of equations $\nabla f = \mathbf{0}$. (Notice this is a system n equations if f is a function of n variables! We must set *each* partial derivative equal to 0.) In general, this system will often be nonlinear, so it may be very difficult to solve for all the critical points! When f is a function of two variables, there are two equations that must be true at a critical point. The strategy for finding all the critical points is to consider one equation at a time, figure out all the possibilities for solving it, then in each case, try to solve the second equation also. Always keep in mind that a critical point (x_o, y_o) needs to simultaneously solve *both* equations!

EXAMPLE 1 Find the critical points of $f(x,y) = x^3 \sin y - x$ and classify them.

We need to solve $f_x(x,y) = 0$ and $f_y(x,y) = 0$:

$$3x^2 \sin y - 1 = 0$$
$$x^3 \cos y = 0$$

The first equation looks very difficult to solve, but the second equation tells us that we must have either $x = 0$ or $\cos y = 0$ at a critical point. Therefore, we consider two cases.

Case I: If $x = 0$. This case gives us no critical points since the first equation becomes $-1 = 0$, which is impossible.

Case II: If $\cos y = 0$, this means $y = n\frac{\pi}{2}$ for some odd integer n. If $n = 4k-1$, $\sin y = -1$ and if $n = 4k-3$, $\sin y = 1$. Therefore, when $n = 4k-1$, we would need to solve $3x^2 + 1 = 0$ to find a critical point, but this is impossible! When $n = 4k-3$, we need to solve $3x^2 - 1 = 0$; this is true when $x = \pm\frac{1}{\sqrt{3}}$. Therefore, for every other odd n, there are *two* critical points.

All the critical points: $(\frac{1}{\sqrt{3}}, \frac{(4k-3)\pi}{2})$ and $(-\frac{1}{\sqrt{3}}, \frac{(4k-3)\pi}{2})$ where k is any integer.

To classify these critical points, we calculate the determinant of the Hessian matrix (at any point (x,y)):

$$|Hf(x,y)| = \begin{vmatrix} f_x x & f_x y \\ f_y y & f_y y \end{vmatrix} = \begin{vmatrix} 6x \sin y & 3x^2 \cos y \\ 3x^2 \cos y & -x^3 \sin y \end{vmatrix} = -6x^4 \sin^2 y - 9x^4 \cos^2 y$$

Plugging in any critical point $(\pm\frac{1}{\sqrt{3}}, \frac{(4k-3)\pi}{2})$, we get that

$$\left|Hf(\pm\tfrac{1}{\sqrt{3}}, \tfrac{(4k-3)\pi}{2})\right| = -\frac{2}{3} < 0,$$

so each critical point must be a saddle point.

This next example is possibly more algebra than you would be expected to do on homework or a test, but it's a good illustration of the logic of solving a system of equations!
EXAMPLE 2 $f(x, y, z) = x^2yz + y^2z + z^3 - 3z$

We need to solve the system of three equations:

$$\begin{gathered} 2xyz = 0 \\ x^2z + 2yz = 0 \\ x^2y + y^2 + 3z^2 - 3 = 0 \end{gathered}$$

The first equation looks the simplest: from it, we see that either $x = 0$, $y = 0$, or $z = 0$.

Case I: $x = 0$.
First, look at the second equation. When $x = 0$, we must have either $y = 0$ or $z = 0$ to make the second equation true. This gives us two sub-cases.... if $y = 0$, the last equation gives us that $z^2 = 1$, or $z = \pm 1$. If $z = 0$, then the last equation tells us that $y^2 = 3$, or $y = \pm\sqrt{3}$. In each sub-case, we found two critical points. In total, the four critical points we found are: $(0, 0, \pm 1)$ and $(0, \pm\sqrt{3}, 0)$.

Case II: $y = 0$.
In this case, the second equation tell us that either $x = 0$, or $z = 0$. The last equation is impossible if both $y = z = 0$. If $y = x = 0$, the last equation gives $z = \pm 1$, so we find two of the critical points that we already found in Case I.

Case III: $z = 0$.
In this case, the second equation is $0 = 0$, which is always true. The third equation becomes $x^2y + y^2 - 3 = 0$. This means that $y = \frac{-x^2 \pm \sqrt{x^4+12}}{2}$ (by the quadratic formula), so here we've found infinitely many critical points in this case: two for each value of x! (*And notice that if $x = 0$, $y = \pm\sqrt{3}$, so we have the critical points $(0, \pm\sqrt{3}, 0)$ that we already found in Case 1.*)

All the critical points: $(0, 0, \pm 1)$ and $(x, \frac{-x^2 \pm \sqrt{x^4+12}}{2})$ for all real numbers x.

Question: Could you re-solve this system, looking at the second equation first? Factoring the second equation, you see that either $z = 0$, or $y = -\frac{x^2}{2}$. What next?

Finding Absolute Minimums and Maximums

The extreme value theorem guarantees that a (continuous) function achieves its maximum and its minimum values on a closed bounded domain. To find the maximum and minimum values, recall that we check the function's values at all of the critical points inside the domain, as well as the function's values on the boundary of the domain. By checking at these points, notice we find the absolute maximum (minimum) *value* – that is, the biggest (smallest) the function can be – as well as the *point(s)* that value is achieved at.

The following examples are from the book, but are rewritten slightly:
Example 4.33 Find the absolute maximum and minimum values of the function $f(x, y) = 2x^2 + 2y^2 - x + y$ on the closed disk $D = \{(x, y) | x^2 + y^2 \leq 1\}$.

First, find the critical points of $f(x, y)$. Calculate that $\nabla f(x, y) = (4x - 1, 4y + 1)$. Setting both $4x - 1 = 0$ and $4y + 1 = 0$, we see we must have both $x = \frac{1}{4}$ and $y = -\frac{1}{4}$. Therefore, the only critical point is $(\frac{1}{4}, -\frac{1}{4})$. This point is in the disk D, so we check the value of $f(\frac{1}{4}, -\frac{1}{4}) = -\frac{1}{4}$.

Now, notice that the boundary of D is all the points on the circle $x^2 + y^2 = 1$. We can break up the boundary into two parts: the upper half-circle $y = \sqrt{1 - x^2}$ and the lower half-circle $y = -\sqrt{1 - x^2}$.

For the upper half-circle, consider $g(x) = 2 - x + \sqrt{1 - x^2}$ (for $-1 \leq x \leq 1$). For all the points on the upper half-circle $y = \sqrt{1 - x^2}$, $g(x) = f(x, y)$. We will find the extreme values of $g(x)$ on the interval $[-1, 1]$. Setting $g'(x) = -1 - x/\sqrt{1 - x^2} = 0$, we find the only critical point is at $x = -\frac{1}{\sqrt{2}}$. We need to check the values of g at this point and at the endpoints of the interval:

$$g(-\tfrac{1}{\sqrt{2}}) = 2 + \sqrt{2}; \quad g(-1) = 3; \quad g(1) = 1;$$

None of these values are smaller than $f(\frac{1}{4}, -\frac{1}{4})$, so none are candidates for the absolute minimum of f. Since we're on the upper half circle, $f(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = g(-\frac{1}{\sqrt{2}}) = 2 + \sqrt{2}$. This is the largest value we've found for f (so far!), so this is a possibility for the absolute maximum of f.

To check the values of f on the lower half-circle, consider $h(x) = f(x, -\sqrt{1 - x^2}) = 2 - x - \sqrt{1 - x^2}$ (for $-1 \leq x \leq 1$). To find the critical values of $h(x)$ on $[-1, 1]$, find the critical points in the interval and check the values of $h(x)$ at these critical points and at the endpoints. You should find that the only critical point is $x = \frac{1}{\sqrt{2}}$ and that $h(x)$ is maximized at $x = -1$ (with value $h(-1) = 3$) and minimized at $x = \frac{1}{\sqrt{2}}$ (with value $h(\frac{1}{\sqrt{2}}) = 2 - \sqrt{2}$). Neither of these values is larger or smaller than ones we've already found, though.

Putting this together, we see that $f(x, y)$ is minimized at $(\frac{1}{4}, -\frac{1}{4})$, with value $f(\frac{1}{4}, \frac{1}{4}) = -\frac{1}{4}$, and is maximized at $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, with value $f(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = 2 + \sqrt{2}$.

Notice that we could have checked the boundary in one step: we can describe all the points on the circle by $(\cos\theta, \sin\theta)$ where θ ranges from 0 to 2π. Then, on the boundary, the function is equal to $g(\theta) = 2(\cos\theta)^2 + 2(\sin\theta)^2 - \cos\theta + \sin\theta = 2 - \cos\theta + \sin\theta$. Then, we find critical points wherever $g'(\theta) = \sin\theta + \cos\theta = 0$. On $[0, 2\pi]$, $\tan\theta = -1$ only at $\frac{3\pi}{4}$ and $\frac{7\pi}{4}$. These values of θ correspond to the points $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ and $(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$ that we found above! Checking the values of the function at these two points on the boundary and at the critical point on the inside of the disk, we find the same absolute minimum and maximum as with the previous method. This is the method the book uses; feel free to use whichever you like better. (*This method is typically easier because you can avoid the square roots in the derivatives and algebra above! It's a good reason to review trig, including learning the values where sine, cosine, and tangent are* 0 *and* ± 1.)

EXAMPLE 4.34 Find the absolute minimum and absolute maximum values of the function $f(x, y) = x^3 - 3xy + 3y^2$ on the rectangle $D = \{(x, y)|\frac{1}{4} \leq x \leq 1, 0 \leq y \leq 2\}$. (I changed the rectangle slightly from what is given in the book.)

First, solve for the critical points of the function: Set $3x^2 - 3y = 0$ and $6y - 3x = 0$. From the first equation, at a critical point, we must have $y = x^2$. Then from the second, $6x^2 - 3x = 0$ implies that either $x = 0$ or $x = \frac{1}{2}$. Therefore, the only two critical points are $(0, 0)$ and $(\frac{1}{2}, \frac{1}{4})$. However, the point $(0, 0)$ is not on the rectangle we're interested in, so we do not consider it! The value of the function at the only critical point that is in the rectangle is $f(\frac{1}{2}, \frac{1}{4}) = -\frac{1}{16}$.

We break the boundary up into four different lines: $x = \frac{1}{4}$ (with $0 \leq y \leq 2$); $x = 1$ (with $0 \leq y \leq 2$); $y = 0$ (with $\frac{1}{4} \leq x \leq 1$); and $y = 2$ (with $\frac{1}{4} \leq x \leq 1$). (Draw a picture to see this clearly!)

On the line $x = \frac{1}{4}$, consider $g(y) = f(\frac{1}{4}, y)$ (on the interval $[0, 2]$). To solve for the critical points, set $g'(y) = 0$: $g'(y) = -\frac{3}{4} + 6y = 0$. The only critical point is at $y = \frac{1}{8}$. Since this is in $[0, 2]$, check the values $g(\frac{1}{8}) = \frac{1}{32}$, $g(0) = \frac{1}{64}$, and $g(2) = \frac{673}{64}$. Notice that the point on this line corresponding to $y = 2$ (that is, the point $(\frac{1}{4}, 2)$) is a possible maximum for f (It's the largest value of f we've found so far, but we still have three more parts of the boundary to check!)

Similarly, on $y = 0$, consider $g(x) = x^3$ on the interval $[\frac{1}{4}, 1]$. Once you've found the maximum and minimum on this line (as well as on the other two lines that make up the boundary), compare all the values you've checked to find out that the absolute maximum and the absolute minimum are

$$\begin{aligned} f(\tfrac{1}{4}, 2) &= \tfrac{673}{64} \quad (\leftarrow \text{ absolute maximum}) \\ f(\tfrac{1}{2}, \tfrac{1}{4}) &= -\tfrac{1}{16} \quad (\leftarrow \text{ absolute minimum}) \end{aligned}$$

EXAMPLE 4.35 Try this example on your own! When you look at the boundary, you can break it into three parts: the upper half-circle $y = \sqrt{1 - x^2}$ (with $-1 \leq x \leq 0$); the line $y = -x + 1$ (with $0 \leq x \leq 1$) and the lower half-circle $y = -\sqrt{1 - x^2}$ (with $-1 \leq x \leq 1$).

Or, if you'd prefer, you can divide the boundary into two parts: the line $y = -x + 1$ (with $0 \leq x \leq 1$) and the circle, described by $(\cos\theta, \sin\theta)$ where $\frac{\pi}{2} \leq \theta \leq 2\pi$.