# Homework #7 (Solns.)

**5.4** #18 $\vec{F} = (3x^2 \ln x + x^2)\hat{i} + x^3 y^{-1} \hat{j}$

$$\vec{\nabla} \times \vec{F} = \text{curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ (3x^2 \ln x + x^2) & (x^3 y^{-1}) & 0 \end{vmatrix}$$

$$= 0\hat{i} + 0\hat{j} + [3x^2 y^{-1} - 0]\hat{k}$$

$$\neq \vec{0}$$

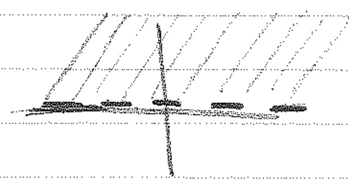
So $\vec{F}$ is not a gradient vector field

one w/ $\vec{F}$ a g.v.f (it wasn't assigned!)

#19. $\vec{F} = 2x \ln y \,\hat{i} + (2y + x^2/y)\hat{j}$

Check that curl $\vec{F} = 0\hat{i} + 0\hat{j} + \left[\frac{2x}{y} - \frac{2x}{y}\right]\hat{k} = \vec{0}$

the domain of $\vec{F}$ is $\{(x,y) : y > 0\}$



(because $\ln y$ is only defined for $y > 0$ (and also $\frac{1}{y}$ is only defined for $y \neq 0$))

To find $f$ with $\vec{\nabla} f = \vec{F}$ we need

$$\frac{\partial f}{\partial x} = 2x \ln y \quad \Rightarrow \quad f(x,y) = x^2 \ln y + c_1(y)$$

and $$\frac{\partial f}{\partial y} = 2y + \frac{x^2}{y} \quad \Rightarrow \quad f(x,y) = y^2 + x^2 \ln y + c_2(x)$$

The only functions that work are

$$\boxed{f(x,y) = x^2 \ln y + y^2 + C}$$

Where C is any constant number.

6.2 #14 $\displaystyle\int_{-1}^{1}\left(\int_0^{3x} e^{x+3y}\,dy\right)dx$

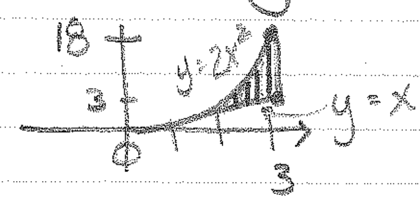
$$= \int_{-1}^{1}\left(\frac{e^{x+3y}}{3}\Big|_{y=0}^{3x}\right)dx = \int_{-1}^{1}\left(\frac{1}{3}e^{10x} - \frac{1}{3}e^{x}\right)dx$$

$$= \frac{1}{30}e^{10x} - \frac{1}{3}e^{x}\Big|_{-1}^{1} = \frac{1}{30}\left[e^{10} - e^{-10} - 10e + 10e^{-1}\right]$$

#22 Evaluate

$\displaystyle\iint_D e^x\,dA$ with $D = \{(x,y) : 0 \le x \le 3,\ x \le y \le 2x^2\}$

$$\int_0^3\left(\int_0^{2x^2} e^x\,dy\right)dx$$




$$= \int_0^3\left(ye^x\Big|_{y=x}^{2x^2} dx\right) = \int_0^3\left(2x^2e^x - xe^x\right)dx$$

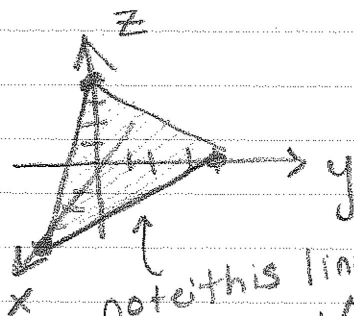
By PARTS $\displaystyle = \int_0^3\left(-4xe^x + e^x\right)dx + \left(2x^2e^x - xe^x\right)\Big|_0^3$

$$= \left[\int_0^3 4e^x\,dx - 4xe^x\Big|_0^3 + e^x\Big|_0^3\right] + \left(18e^3 - 3e^3 - 0\right)$$

$$4e^x\Big|_0^3 - 12e^3 + (e^3-1) + 15e^3$$

$$= \boxed{8e^3 - 5}$$

**6.3 #7** Draw the four planes




note: this line is $x+y+0=4$ or $y=4-x$

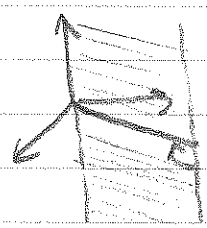
$\underline{x+y+z=4}$

has x-intercept $=4$

y-inter. $=4$

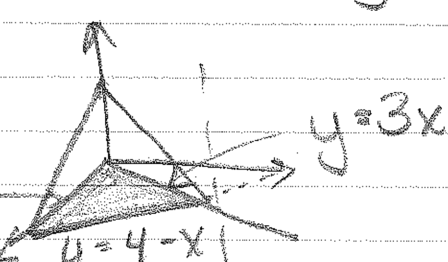
z-inter. $=4$

the plane $y=3x$ ($z$ can have any value) is



$y=3x$ (in $z=0$ plane)

The volume is always below $x+y+z=4$ (or $z=4-x-y$) and above the triangle in the x-y plane:




defined by

$y=0$

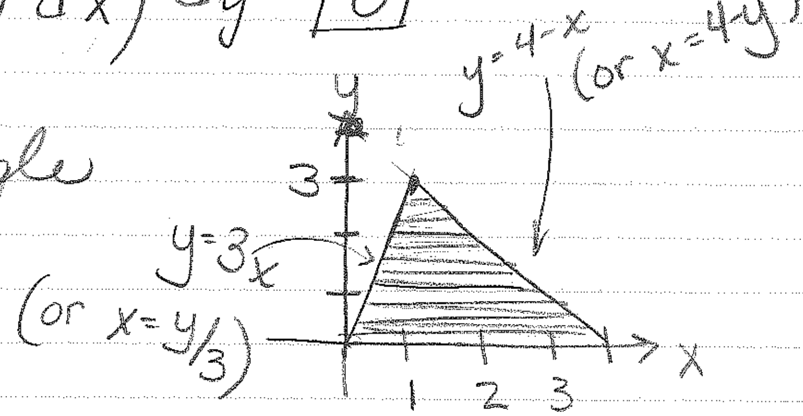
$y=4-x$

$y=3x$.

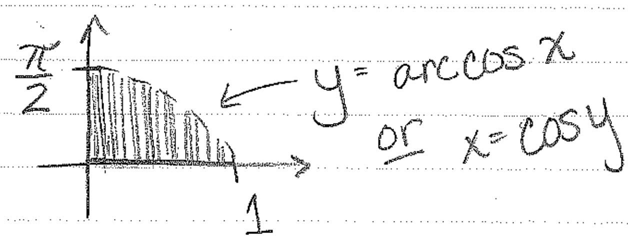
Therefore, compute the integral

$$\int_0^3 \left( \int_{y/3}^{4-y} (4-x-y)\, dx \right) dy = \boxed{8}$$

over the triangle




#17 $\int_0^1 \int_0^{\arccos x} x \; dy\, dx$




Switching the order, we have $0 \le x \le \cos y$

for each $y$ from $0$ to $\pi/2$ :

$$\int_0^{\pi/2} \left[ \int_0^{\cos y} x\, dx \right] dy = \int_0^{\pi/2} \frac{\cos^2 y}{2}\, dy$$

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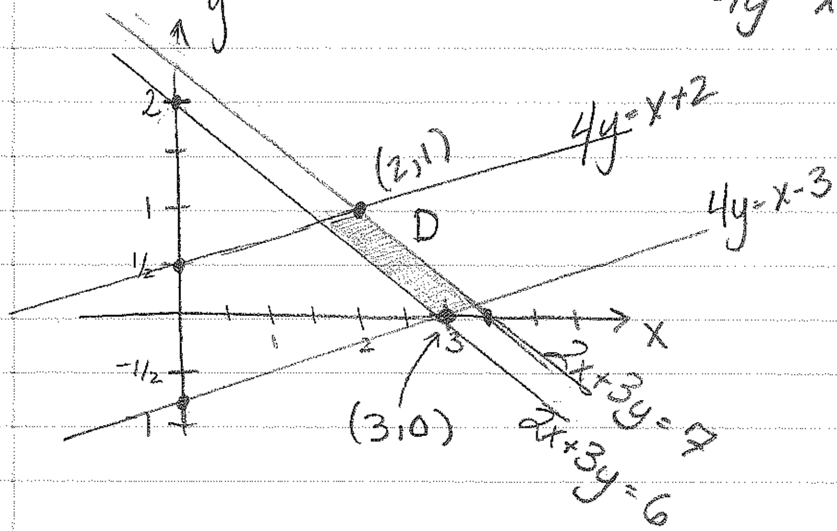
$$= \frac{1}{4} \int_0^{\pi/2} (1 + \cos 2y)\, dy = \boxed{\frac{\pi}{8}}$$

6.4 #22 Compute $\iint_D (4x+6y)$ where $D$ is the region bounded by $4y = x-3$, $2x+3y=6$, $4y = x+2$, $2x+3y=7$




We want to use the change of variables

$$u = 2x+3y$$
$$v = 4y - x$$

because the given lines will become horizontal and vertical ones! ($u=6$, $u=7$, $v=-3$ and $v=2$)

[Notice this is, up to a constant, the change of variables given in the book – they give $x$ and $y$ as functions of $u$ and $v$, but you can solve those 2 equations for $u$ and $v$.]

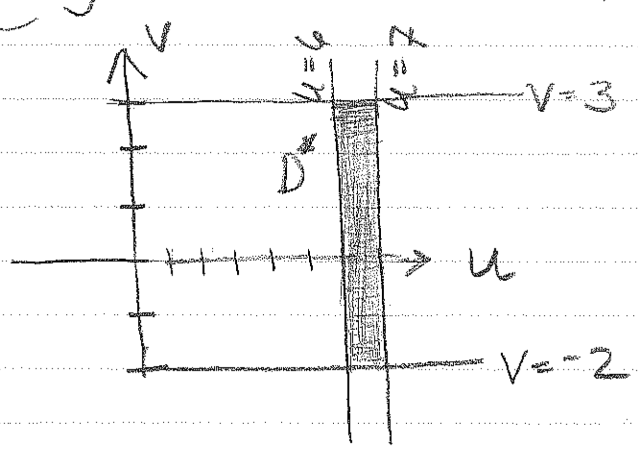
Aside: You could also find all the corners of the parallelogram $D$ (by solving for the points of intersection of the lines) ~ they are

$$(x,y) = (3,0),\ \left(\frac{18}{11}, \frac{10}{11}\right),\ (2,1),\ \text{and}\ \left(\frac{37}{11}, \frac{1}{11}\right)$$

These points map to (in the $(u,v)$ plane)

$(u,v) = (6,-3), (6,2), (7,2)$, and $(7,-3)$

Notice this work wasn't necessary though! We already know that $D$ becomes the box with sides $u=6$, $u=7$, $v=-3$, $v=2$, so we alredy knew the corners of this box.




Notice $f(x,y) = 4x + 6y$ becomes

$f(u,v) = 2 \cdot u$.

Finally $\dfrac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ -1 & 4 \end{vmatrix} = +11$

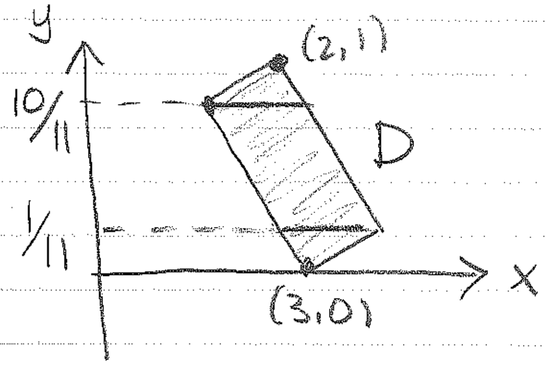
So $du\,dv = \left| \dfrac{\partial(u,v)}{\partial(x,y)} \right| dx\,dy = 11\,dx\,dy$

$$\iint_D (4x+6y)\,dx\,dy = \int_{-2}^{3} \int_{6}^{7} \frac{2u}{11}\,du\,dv = (3-(-2))\frac{1}{11}(7^2-6^2)$$

$$= \boxed{\frac{65}{11}}$$

The change of variables worked out very quickly for this example, and the advantage of integrating over a rectangle made it worth computing the Jacobian and the new domain!

You could have directly integrated over D, however it's a lot more work! (It would require finding the corner points, needed to separate the region into three parts.)




$$\iint_D (4x+6y)\,dA = \int_0^{1/11}\left(\int_{\frac{6-3y}{2}}^{4y+3}(4x+6y)\,dx\right)dy$$

$$+ \int_{1/11}^{10/11}\left(\int_{\frac{6-3y}{2}}^{\frac{7-3y}{2}}(4x+6y)\,dx\right)dy$$

$$+ \int_{10/11}^{1}\left(\int_{4y-2}^{\frac{7-3y}{2}}(4x+6y)\,dx\right)dy$$

$= \ldots$ should work out to the same answer, $\frac{65}{11}$, of course!