Problems from 6.5, Ch 7, 8.1

6.5 #10 $\int_0^2 \int_0^x \int_0^3 xy\,dz\,dy\,dx$

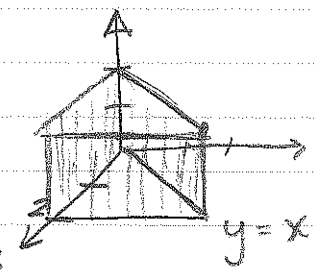
$$= \int_0^2 \int_0^x (xyz)\Big|_{z=0}^{3} dy\,dx = \int_0^2 \int_0^x 3xy\,dy\,dx$$

$$= \int_0^2 \tfrac{3}{2} xy^2\Big|_{y=0}^{x} dx = \int_0^2 \frac{3}{2} x^3 dx = \frac{3}{8} x^4\Big|_0^2$$

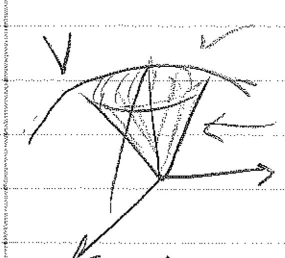
$$= \boxed{6}$$

The volume we integrated over had, for each $x \in [0,2]$, $0 \le y \le x$ and $0 \le z \le 3$

Picture: a wedge-shaped region



#20 Find the volume inside ① $z^2 = x^2+y^2$
and ② $x^2+y^2+z^2=1$
and ③ above the xy-plane

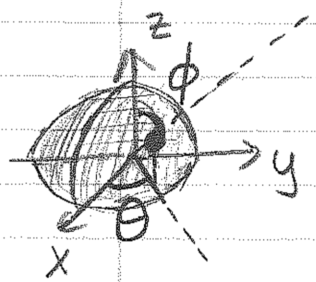


Using spherical coordinates: ① (& ③) $\Rightarrow 0 \le \phi \le \pi/4$
② $\Rightarrow 0 \le \rho \le 1$

$$\iiint_V dV = \iiint_V dx\,dy\,dz = \iiint_V \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta.$$

$$= \int_0^{2\pi} \int_0^{\pi/4} \int_0^1 \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta = \boxed{2\pi\left(1-\frac{1}{\sqrt{2}}\right)\frac{1}{3}}$$

7.1 #4 The quarter-sphere $x^2+y^2+z^2=a^2$ with $z \geq 0$ and $x \geq 0$ is parameterized using spherical coordinates with



$$z = a\sin\phi \geq 0 \Rightarrow 0 \leq \phi \leq \frac{\pi}{2}$$

and $x = a\cos\theta\cos\phi \geq 0 \Rightarrow -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.

$$\boxed{\begin{array}{l}\vec{r}(\theta,\phi) = (a\cos\theta\cos\phi,\ a\sin\theta\sin\phi,\ a\sin\phi) \\ \quad\text{with } \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}] \text{ and } \phi \in [0, \pi/2]\end{array}}$$

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$$\vec{r}(u,v) = (u, e^u\sin v, e^u\cos v) \qquad \begin{array}{l} 0 \leq v \leq 2\pi \\ -\infty < u < \infty\end{array}$$

$$\vec{T}_u = \frac{\partial \vec{r}}{\partial u} = (1, e^u\sin v, e^u\cos v)$$

$$\vec{T}_v = \frac{\partial \vec{r}}{\partial v} = (0, e^u\cos v, -e^u\sin v)$$

$$\vec{N}(u,v) = \vec{T}_u \times \vec{T}_v = \boxed{-e^{2u}\,\hat{\imath} + e^u\sin v\,\hat{\jmath} + e^u\cos v\,\hat{k}}$$

7.3 #6 Calculate $\vec{N} = \cos u\,\hat{\imath} + \sin u\,\hat{\jmath}$

$$\|\vec{N}\| = \sqrt{\cos^2 u + \sin^2 u} = 1.$$

$$\iint_S 2z(x^2+y^2)\,dS = \int_0^{\pi}\left(\int_0^2 2v\underbrace{(\cos^2 u + \sin^2 u)}_{=1}\underbrace{\|\vec{N}\|}_{=1}\,dv\right)du$$

$$= \pi\, v^2\Big|_{v=0}^{2} = \boxed{4\pi}$$

8.1 #6 $\int_{\vec{C}} \left((2-y^3)\hat{\imath} + (y+x^3+2)\hat{\jmath}\right) \cdot d\vec{s}$ $\vec{C}$ is the counter-clockwise circle of radius 5 centered at $(0,0)$

$$= \int_{\vec{C}} (2-y^3)\,dx + (y+x^3+2)\,dy$$

$$= \iint_C \left(\frac{\partial}{\partial x}(y+x^3+2) - \frac{\partial}{\partial y}(2-y^3)\right) dx\,dy \quad \left(\text{Green's } \underline{\text{Thm}}\right)$$

$$= \iint_C (3x^2+3y^2)\,dx\,dy$$

$$= \int_0^{2\pi}\int_0^5 3r^2\, r\,dr\,d\theta \qquad \text{changing variables to polar coords.}$$

$$= \int_0^{2\pi}\int_0^5 3r^3\,dr\,d\theta = 2\pi\, \frac{3r^4}{4}\Big|_0^5 = \boxed{\frac{3\cdot 5^4\pi}{2}}$$

#10 $\int_{\vec{C}} (2x+3y+2)\,dx - (x-4y+3)\,dy$ $\left[\text{Recall } \int_{\vec{C}} \vec{F}\cdot ds = -\int_{-\vec{C}} \vec{F}\cdot d\vec{s}\right]$

$$= -\int_{-\vec{C}} \left((2x+3y+2)\,dx - (x-4y+3)\,dy\right)$$

To use Green's Thm we must integrate along $-\vec{C}$, which is oriented <u>counter</u> clockwise

$$= \iint_D \left[\frac{\partial}{\partial x}(x-4y+3) - \frac{\partial}{\partial y}\left(-(2x+3y+2)\right)\right] dx\,dy = \iint_D 4\,dx\,dy$$

$= 4(2\pi) = \boxed{8\pi}$ since area $D = (2)(1)\pi = 2\pi$

Notice you could also compute the line integral directly:

Use the parameterization

$$\vec{c}(\theta) = (2\cos\theta, -\sin\theta) \quad \theta \in [0, 2\pi]$$

(this goes clockwise around the ellipse)

$$\int_{\vec{c}} \vec{F} \cdot d\vec{s} = \int_0^{2\pi} \Big[(4\cos\theta - 3\sin\theta + 2)(-2\sin\theta) + (2\cos\theta - 4\sin\theta + 3)(\cos\theta)\Big] d\theta$$

$$= 8\pi$$