Math 5B: Answers to Midterm Review Problems

1. (a) \( \lim_{(x,y) \to (0,2)} e^{\frac{-x^2}{y^2}} = \sqrt[4]{e} \)

(b) \( \lim_{(x,y) \to (1,0)} \frac{xy(y-x+1)}{(x-1)^2+y^2} \) does not exist. (Check the limit along the lines \( x = 1 \) and \( y = 0 \). Use good notation when computing your limits!)

2. The level curves of \( f(x, y) \) are the ellipses \( x^2 + 4y^2 = c \) for \( c \geq 0 \) (there are no level curves for \( c < 0 \)). The unit vector in the direction \( \mathbf{v} \) is \( \mathbf{u} = (\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}) \). \( D_u f(2,0) = \frac{4}{\sqrt{5}} \) and \( D_u f(\sqrt{2}, \frac{1}{\sqrt{2}}) = 2\sqrt{10} \).

3. Using the point \((0,1): df = \Delta x \). Then \( \Delta f = f(0,0.2) - f(0,1) \approx .9 \) so \( f(0.9,0.2) \approx 0.9 \). Using the point \((1,0) \) should give a better approximation, though! At \((1,0), df = \Delta x + \Delta y \). Then \( f(0.9,0.2) \approx f(1,0) + (0.9 - 1) + (0.2 - 0) = 1.1 \).

4. Show \( f_{xx} + f_{yy} = 0 \). (Use the chain rule and the results from implicit differentiation of \( x = r \cos \theta \) and \( y = r \sin \theta \): \( \frac{\partial r}{\partial x} = \cos \theta, \frac{\partial \theta}{\partial x} = -\frac{\sin \theta}{r} \), etc. OR, substitute to get \( f(x,y) = (x^2+y^2) \sin(2 \arctan \frac{y}{x}) \).)

5. By Taylor’s formula \( f(x,y) \approx x + x(y - 1) = xy \).

5. Using the implicit function theorem, we can locally solve for \( u(x,y) \) and \( v(x,y) \) near a point \((x_0,y_0,u_0,v_0)\) satisfying both of the given equations as long as

\[
\begin{vmatrix}
2u_0 & 2v_0 \\
2u_0 & 4v_0 \\
\end{vmatrix} = 4u_0v_0 \neq 0.
\]

In other words, whenever both \( u_0 \neq 0 \) and \( v_0 \neq 0 \). Implicitly differentiating both equations, we find first that \( \frac{\partial u}{\partial x} = \frac{e^x+x}{u} \), then \( \frac{\partial^2 u}{\partial x^2} = \frac{e^x+1}{u} - \frac{(e^x+x)^2}{u^3} \).

6. \( y-z = 0 \)

7. The minimum temperature is \(-1\) (it occurs at the critical point \((1,0)\)).

The maximum temperature is \(5\) (it occurs at the boundary points \((-1,1)\) and \((-1,-1)\)).

8. There are two critical points: \((0,0)\) is a saddle point and \((6,18)\) is a local minimum.

9. \( \text{div} \mathbf{F} = \frac{1}{x} + \frac{x}{\sqrt{1-(xy)^2}} \); \( \text{curl} \mathbf{F} = \left( \frac{3}{2}x \sqrt{y-x} \right) \hat{i} + \left( \frac{1}{z} - y^2 \right) \hat{j} + \left( \frac{y}{\sqrt{1-(xy)^2}} + e^z \right) \hat{k} \)

10. \#1 and \#4 are meaningless; \#2, \#3, \#6 are vector fields; \#5 is a scalar function.