Math 5B: Answers to Midterm Review Problems

1. (a) $\lim_{(x,y)\to(0,2)} e^{-\frac{x-1}{y^2}} = \sqrt[4]{e}$

(b) $\lim_{(x,y)\to(1,0)} \frac{xy(y-x+1)}{(x-1)^2+y^2}$ does not exist. (Check the limit along the lines $x = 1$ and $y = 0$. Use good notation when computing your limits!)

2. The level curves of $f(x, y)$ are the ellipses $x^2 + 4y^2 = c$ for $c \geq 0$ (there are no level curves for $c < 0$). The unit vector in the direction $\mathbf{v}$ is $\mathbf{u} = (\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}})$. $D_{\mathbf{u}}f(2,0) = \frac{4}{\sqrt{5}}$ and $D_{\mathbf{u}}f(\sqrt{2}, \frac{1}{\sqrt{2}}) = 2\sqrt{10}$.

3. Using the point $(0, 1)$: $df = \Delta x$. Then $\Delta f = f(0.9, 0.2) - f(0, 1) \approx .9$ so $f(0.9, 0.2) \approx 0.9$. Using the point $(1, 0)$ should give a better approximation, though! At $(1, 0)$, $df = \Delta x + \Delta y$. Then $f(0.9, 0.2) \approx f(1, 0) + (0.9 - 1) + (0.2 - 0) = 1.1$.

4. Show $f_{xx} + f_{yy} = 0$. (Use the chain rule and the results from implicit differentiation of $x = r\cos\theta$ and $y = r\sin\theta$: $\frac{\partial r}{\partial x} = \cos\theta$, $\frac{\partial \theta}{\partial x} = \frac{-\sin\theta}{r}$, etc. OR, substitute to get $f(x, y) = (x^2 + y^2)\sin(2\arctan\frac{y}{x})$.)

5. By Taylor's formula $f(x, y) \approx x + x(y - 1) = xy$.

5. Using the implicit function theorem, we can locally solve for $u(x, y)$ and $v(x, y)$ near a point (x_o, y_o, u_o, v_o) satisfying both of the given equations as long as

$$\begin{vmatrix} 2u_o & 2v_o \\ 2u_o & 4v_o \end{vmatrix} = 4u_o v_o \neq 0.$$

In other words, whenever both $u_o \neq 0$ and $v_o \neq 0$. Implicitly differentiating both equations, we find first that $\frac{\partial u}{\partial x} = \frac{e^x + x}{u}$, then $\frac{\partial^2 u}{\partial x^2} = \frac{e^x + 1}{u} - \frac{(e^x + x)^2}{u^3}$.

6. $y - z = 0$

7. The minimum temperature is -1 (it occurs at the critical point $(1, 0)$).
The maximum temperature is 5 (it occurs at the boundary points $(-1, 1)$ and $(-1, -1)$).

8. There are two critical points: $(0, 0)$ is a saddle point and $(6, 18)$ is a local minimum.

9. div $\mathbf{F} = \frac{1}{x} + \frac{x}{\sqrt{1-(xy)^2}}$; curl $\mathbf{F} = \left(\frac{3}{2}x\sqrt{y} - xe^z\right)\mathbf{i} + \left(\frac{1}{z} - y^{\frac{3}{2}}\right)\mathbf{j} + \left(\frac{y}{\sqrt{1-(xy)^2}} + e^z\right)\mathbf{k}$

10. #1 and #4 are meaningless; #2, #3, #6 are vector fields; #5 is a scalar function.