Midterm
Math 5B, Fall 2010

You have 1 hour and 15 minutes to complete this exam.

Name: Solutions

Perm #: ____________

Signature: ____________

Discussion section: 1 2 3 4 5 6

Please write clearly and show all your work. Partial credit will be given only if your work is relevant and correct.

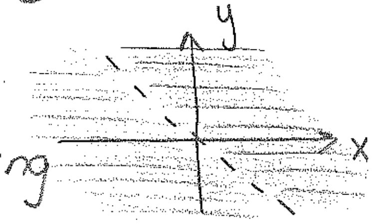
Problem	Possible Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
Total	50	

Exercise 1. Consider the function $f(x, y) = \dfrac{2x + y}{x + y}$.

(a) Describe the domain of the function and sketch a picture of it in the x-y plane.

The denominator is 0 whenever $x+y=0$

$$\boxed{\text{Domain} = \{(x,y) \in \mathbb{R}^2 : x+y \neq 0\}}$$



Sketch: include everything excep the dotted line $y=-x$

(b) Show that $\lim\limits_{(x,y)\to(0,0)} f(x, y)$ does not exist.

Show the limits along 2 different lines going through (0,0) give different results:

$$\lim_{\substack{(x,y)\to(0,0) \\ \text{along } x=0}} \frac{2x+y}{x+y} = \lim_{y\to 0} \frac{y}{y} = \lim_{y \to 0} 1 = \underline{1}$$

$$\lim_{\substack{(x,y)\to(0,0) \\ \text{along } y=0}} \frac{2x+y}{x+y} = \lim_{x\to 0} \frac{2x}{x} = \lim_{x\to 0} 2 = 2$$

(c) Describe all of the level curves of the function $f(x, y)$. Draw a contour diagram of some of the level curves.

For each c, find all points (in the domain) where $f(x,y)=c$

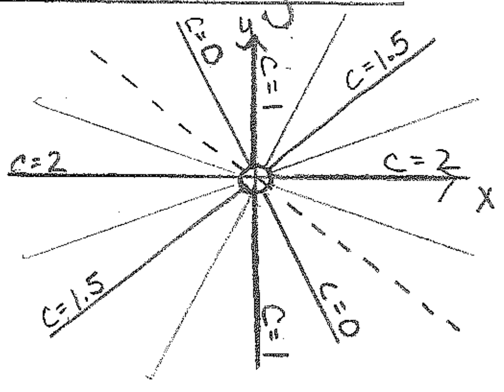
$$c = \frac{2x+y}{x+y} \iff cx+cy = 2x+y$$

$$\iff (c-2)x = (1-c)\,y.$$

If $c=1$, the level curve is $\underline{\underline{x=0}}$ (not including the point (0,0))

If $c \neq 1$, the level curve is the line $\underline{\underline{y = \left(\frac{c-2}{1-c}\right)x}}$ (not including the point (0,0) since its not in the domain.)

Contour diagram:



$c=1$: $\underline{x=0}$

$c=2$: $\underline{y=0}$

$c=0$: $\underline{y=-2x}$

$c=1.5$: $\underline{y=x}$

Notice the level curves do not intersect and they fill in the domain (because the function has a value at each point in the domain!)

Exercise 2. Near what points (x_o, y_o, u_o, v_o) can we locally solve for u and v as functions of x and y?

$$u^2 \cos x + v^2 = 0$$
$$u^2 + v^2 = \sin y$$

Find $\dfrac{\partial u}{\partial x}$ and $\dfrac{\partial v}{\partial x}$.

Let $F_1(x,y,u,v) = u^2 \cos x + v^2$; $\frac{\partial F_1}{\partial u} = 2u\cos x$, $\frac{\partial F_1}{\partial v} = 2v$

$F_2(x,y,u,v) = u^2 + v^2 - \sin y$; $\frac{\partial F_2}{\partial u} = 2u$, $\frac{\partial F_2}{\partial v} = 2v$

We can locally solve $F_1(x,y,u,v) = 0$ and $F_2(x,y,u,v) = 0$ for u and v near points (x_o, y_o, u_o, v_o) where

$$\begin{vmatrix} 2u_o \cos x_o & 2v_o \\ 2u_o & 2v_o \end{vmatrix} = 4u_o v_o \cos x_o - 4u_o v_o \neq 0$$

That is at points where $u_o \neq 0$, $v_o \neq 0$ and $\cos x_o \neq +1$ (notice this means $x_o \neq 2\pi k$ where k is any integer)

Near such a point, we know $u(x,y)$ and $v(x,y)$ are functions of x and y and (x,y,u,v) satisfy the two given equations. Differentiating:

$$\frac{\partial}{\partial x}(u^2 \cos x + v^2) = \frac{\partial}{\partial x}(0) \quad \text{and} \quad \frac{\partial}{\partial x}(u^2 + v^2) = \frac{\partial}{\partial x}(\sin y)$$

$$① \; 2u\frac{\partial u}{\partial x}\cos x - u^2 \sin x + 2v\frac{\partial v}{\partial x} = 0 \quad \text{and} \quad ② \; 2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} = 0$$

We can solve these two equations for $\partial u/\partial x$ and $\frac{\partial v}{\partial x}$

Subtracting ① - ②

$$(2u\cos x - 2u)\frac{\partial u}{\partial x} = u^2 \sin x \Rightarrow \boxed{\frac{\partial u}{\partial x} = \frac{u \sin x}{2(\cos x - 1)}}$$

From ②, $\frac{\partial v}{\partial x} = -\frac{u}{v}\frac{\partial u}{\partial x} = \boxed{\frac{u^2 \sin x}{2v(1-\cos x)} = \frac{\partial v}{\partial x}}$

Exercise 3. Find the tangent plane to the surface $z = xy^4 + x^2y^2 - 2 + x$ at the point (2, 1, 6).

This surface is the graph of $z = f(x,y)$
where $f(x,y) = xy^4 + x^2y^2 - 2 + x$.
(Notice $(2,1,6)$ is a point on the surface: $6 = f(2,1)$)

The equation that describes the surface of the tangent plane is

$$z = f(2,1) + f_x(2,1)(x-2) + f_y(2,1)(y-1)$$

$$f_x(x,y) = y^4 + 2xy + 1 \Rightarrow f_x(2,1) = 1^4 + 2\cdot 2\cdot 1 + 1 = 6$$
$$f_y(x,y) = 4xy^3 + 2x^2y \Rightarrow f_y(2,1) = 4\cdot 2\cdot 1^3 + 2\cdot 2^2\cdot 1 = 16$$

$$z = 6 + 6(x-2) + 16(y-1)$$
$$\boxed{z = 6x + 16y - 22}$$

Equivalently, the linear approximation of $f(x,y)$ at $(2,1)$ is $g(x,y) = 6x + 16y - 22$ (The graph of $z = g(x,y)$ is the tangent plane.)

Note:

Many people found the normal vector to the plane
Letting $F(x,y,z) = xy^4 + x^2y^2 - 2 + x - z$, the normal to the tangent plane to the surface $F(x,y,z) = 0$ at $(2,1,6)$ is $\vec{n} = \vec{\nabla} F(2,1,6) = (6, 16, -1)$
Since the point $(2,1,6)$ must be on the plane, the equation for the plane is

$$\vec{n} \cdot (x-2, y-1, z-6) = 0$$

$$\Rightarrow 6(x-2) + 16(y-1) - (z-6) = 0$$

or $6x + 16y - z - 22 = 0$, as above

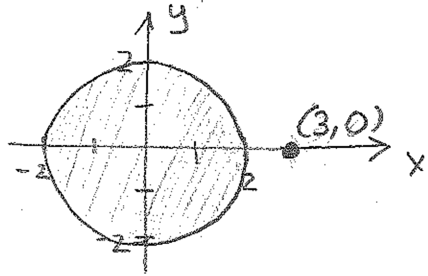
Exercise 4. Find the global maximum and the global minimum of the function $f(x, y) = 3x^2 - 18x + y^2$ on the closed disk $x^2 + y^2 \le 4$.

First, find critical points in the disk.

Set $\vec{\nabla} f(x,y) = (6x-18,\ 2y) = (0,0)$

At a critical point, we must have $x=3$ and $y=0$ to solve this equation. Therefore, the only critical point is $(3,0)$.

So, there are no critical points in the disk of radius 2.



(And, therefore, no local min/max inside the disk)

Now, find the min/max of $f(x,y)$ on the boundary $x^2+y^2=4$

On the boundary $y^2 = 4-x^2$ (for $-2 \le x \le 2$)

and $f(x,y) = g(x) = 3x^2 - 18x + 4 - x^2$

$= 2x^2 - 18x + 4$

To find the max/min of $g(x)$ on $[-2,2]$ first find the critical points where $g'(x)=0$.

$g'(x) = 4x - 18 = 0 \Rightarrow x = 9/2$

but this is not in the interval!

Therefore the max and min of g must occur at the end points: $g(2) = -24$

$g(-2) = 48$

The absolute max and min occur on the boundary, at $(-2,0)$ and $(2,0)$, respectively: $f(-2,0)=48 \leftarrow$ MAX

$f(2,0)=-24 \leftarrow$ MIN

Exercise 5. Find all of the critical points of the function $f(x, y) = xy^2 + x^2 - 4x$ and classify them as local minima, local maxima, or saddle points.

At critical points $\vec{\nabla} f(x,y) = (0,0)$

Solve for all points where both

$$(*)\begin{cases} \dfrac{\partial f}{\partial x} = y^2 + 2x - 4 = 0 \\ \dfrac{\partial f}{\partial y} = 2xy = 0 \end{cases}$$

To make the second equation true, either $x = 0$ OR $y = 0$. When $x = 0$, the first equation gives $y^2 - 4 = 0$, so $y = \pm 2$.
When $y = 0$, the first equation is $2x - 4 = 0$, so $x = 2$.

The points $(0,2)$, $(0,-2)$, $(2,0)$ are the only three points that solve both equations in $(*)$.

To find out what types of critical points these are, use the 2nd derivative test.

$$Hf(x,y) = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} 2 & 2y \\ 2y & 2x \end{bmatrix}$$

$$|Hf(x,y)| = 4x - 4y^2$$

@ $(0,2)$: $|Hf(0,2)| = -16 < 0 \Rightarrow$ $\boxed{(0,2) \text{ is a saddle pt.}}$

@ $(0,-2)$: $|Hf(0,-2)| = -16 < 0 \Rightarrow$ $\boxed{(0,-2) \text{ is a saddle pt.}}$

@ $(2,0)$: $|Hf(2,0)| = 8 > 0$
and $f_{xx}(2,0) = 2 > 0$ $\Rightarrow$ $\boxed{(2,0) \text{ is a local min}}$